Math/CSE 1019C:
Discrete Mathematics for Computer Science
Fall 2012

Jessie Zhao
jessie@cse.yorku.ca

Course page:
http://www.cse.yorku.ca/course/1019

- No more TA office hours
- My office hours will be the same
  - Monday 2-4pm
- Solutions for Test 3 is available online.
- Check your previous test and assignment marks on line
  - By the last four digits of your student ID
  - Available till Dec 10^{th}.
- Assignment 7 will be available to pick up during my office hours
- Final Exam:
  - Coverage: include all materials
  - Closed book exam

- Recall: P(n,r) vs C(n,r)
- C(n,r) is also called binomial coefficient.
- How many bit strings of length 10 contain
  - exactly four 1s?
    C(10,4)=210
  - at most three 1s?
    C(10,0)+C(10,1)+C(10,2)+C(10,3)=176
  - at least 4 1s?
    2^{10}-176=848
  - an equal number of 0s and 1s?
    C(10,5)=252

Binomial Coefficients

$$(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j \quad \text{for } n \geq 0$$

- C(n,r) occurs as coefficients in the expansion of $(a+b)^n$
- Combinatorial proof: refer to textbook

Binomial Coefficients

$$(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j \quad \text{for } n \geq 0$$

- Examples:
  - What is the expansion of $(x+y)^4$?
    $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
  - What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?
    $-(25!*2^{12}*3^{13})/(13!12!)$

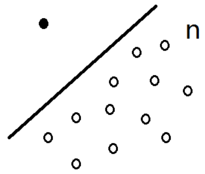
Corollary

$$\sum_{j=0}^{n} C(n,j) = 2^n \quad \text{for } n \geq 0$$

- Proof: Use the Binomial Theorem with x=y=1

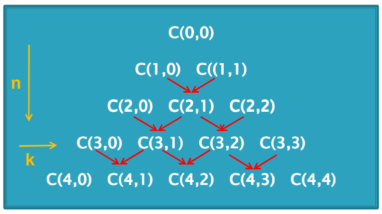
Pascal's Identity

$C(n+1,k) = C(n,k-1) + C(n,k)$ for $1 \leq k \leq n$



Total number of subsets = number including ● + number not including ●
$C(n+1,k) = C(n,k-1) + C(n,k)$

Pascal's triangle



Recurrence Relations

- An easy counting problem: How many bit strings of length n have exactly three zeros?
- A more difficult counting problem: How many bit strings of length n contain three consecutive zeros?

- A recurrence relation (sometimes called a difference equation) is an ***equation*** which defines the nth term in the sequence in terms of (one ore more) previous terms
- A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation
- Recursive definition vs. recurrence relation?

Modeling with Recurrence Relations

- Examples:
  - Fibonacci sequence: $a_n = a_{n-1} + a_{n-2}$
  - Pascal's identity: $C(n+1,k)=C(n,k)+C(n,k-1)$
- Normally there are infinitely many sequences which satisfy the equation. These solutions are distinguished by the initial conditions.

Example 1 – Easy

- Suppose the interest is compounded at 11% annually. If we deposit $10,000 and do not withdraw the interest, find the total amount invested after 30 years.
  - Recurrence relation: $P_n = P_{n-1} + 0.11P_{n-1}$
  - Initial condition: $a_0 = 10{,}000$
  - Answer: $P_{30} = 10000x(1.11)^{30}$

Example 2 (harder)

- Find a recurrence relation for the number of bit strings of length n that do not have two consecutive 0s.
  - a_n : # strings of length n that do not have two consecutive 0s.
  - Case 1: # strings of length n ending with 1 -- a_{n-1}
  - Case 2: # strings of length n ending with 10 -- a_{n-2}
  - This yields the recurrence relation $a_n=a_{n-1}+a_{n-2}$ for $n\geq 3$
  - Initial conditions: $a_1=2$, $a_2=3$

Example 3 (much harder)

- Find a recurrence relation for the number of bit strings of length n which contain 3 consecutive 0s.
- Let S be the set of strings with 3 consecutive 0s. First define the set inductively.
  - Basis: 000 is in S
  - Induction (1): if $w\in S$, $u\in\{0,1\}^*$, $v\in\{0,1\}^*$ then
  - $uwv\in S$

Adequate to define S but NOT for counting. DO NOT count the same string twice.

Example 3 (much harder)

- Find a recurrence relation for the number of bit strings of length n which contain 3 consecutive 0s.
- Let S be the set of strings with 3 consecutive 0s. First define the set inductively.
  - Induction (2): if $w\in S$, $u\in\{0,1\}^*$, then
  - $1w\in S$, $01w\in S$, $001w\in S$, $000u\in S$
- This yields the recurrence

$$a_n=a_{n-1}+a_{n-2}+a_{n-3}+2^{n-3}$$

- Initial conditions: $a_3=1, a_4=3, a_5=8$

Solving Linear Recurrence Relations

- Solve recurrence relations: find a non-recursive formula for $\{a_n\}$
- Easy: for $a_n=2a_{n-1}$, $a_0=1$, the solution is $a_n=2^n$ (back substitute)
- Difficult: for $a_n=a_{n-1}+a_{n-2}$, $a_0=0$, $a_1=1$, how to find a solution?

- Linear Homogeneous Recurrence Relations of degree k with constant coefficients
  - Solving a recurrence relation can be very difficult unless the recurrence equation has a special form

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k},$$
$$\text{where } c_1, c_2, \ldots c_k\in R \text{ and } c_k\neq 0$$

  - Single variable: n
  - Linear: no a_ia_j, a_i^2, a_i^3...
  - Constant coefficients: $c_i\in R$
  - Homogeneous: all terms are multiples of the a_is
  - Degree k: $c_k\neq 0$

Solution Procedure

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k} \quad \text{where } c_1, c_2, \ldots c_k\in R \text{ and } c_k\neq 0$$

- 1. Put all a_i's on LHS of the equation:
  $a_n-c_1a_{n-1}-c_2a_{n-2}-\ldots-c_ka_{n-k}=0$
- 2. Assume solutions of the form $a_n=r^n$, where r is a constant
- 3. Substitute the solution into the equation:
  $r^n-c_1r^{n-1}-c_2r^{n-2}-\ldots-c_kr^{n-k}=0$. Factor out the lowest power of r:
  $r^k-c_1r^{k-1}-c_2r^{k-2}-\ldots-c_k=0$ (called the **characteristic equation**)
- 4. Find the k solutions $r_1, r_2, \ldots, r_k$ of the characteristic equation (characteristic roots of the recurrence relation)
- 5. If the roots are distinct, the general solution is
  $a_n=\alpha_1r_1^n+\alpha_2r_2^n+\ldots+\alpha_kr_k^n$
- 6. The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ are found by enforcing the initial conditions

Example: Solve $a_{n+2}=3a_{n+1}$, $a_0=4$

- $a_{n+2}-3a_{n+1}=0$
- $r^{n+2}-3r^{n-1}=0$, i.e. $r-3=0$
- Find the root of the characteristic equation $r_1=3$
- Compute the general solution $a_n=\alpha_1 r_1^{\,n}=\alpha_1 3^n$
- Find α_1 based on the initial conditions: $a_0=\alpha_1(3^0)$. Then $\alpha_1=4$.
- Produce the solution: $a_n=4(3^n)$

Example: Solve $a_n=3a_{n-2}$, $a_0=a_1=1$

- $a_n-3a_{n-2}=0$
- $r^n-3r^{n-2}=0=0$, i.e. $r^2-3=0$
- Find the root of the characteristic equation $r_1=\sqrt{3}$, $r_2=-\sqrt{3}$
- Compute the general solution

  $a_n=\alpha_1(\sqrt{3})^n+\alpha_2(-\sqrt{3})^n$
- Find α_1 and α_2 based on the initial conditions:

  $a_0=\alpha_1(\sqrt{3})^0+\alpha_2(-\sqrt{3})^0=\alpha_1+\alpha_2=1$

  $a_1=\alpha_1(\sqrt{3})^1+\alpha_2(-\sqrt{3})^1=\sqrt{3}\,\alpha_1-\sqrt{3}\,\alpha_2=1$
- Solution: $a_n=(1/2+1/2\sqrt{3})(\sqrt{3})^n+(1/2-1/2\sqrt{3})(-\sqrt{3})^n$

Example: Find an explicit formula for the Fibonacci numbers

- $f_n-f_{n-1}-f_{n-2}=0$
- $r^n-r^{n-1}-r^{n-2}=0$, i.e. $r^2-r-1=0$
- Find the root of the characteristic equation

  $r_1=(1+\sqrt{5})/2$, $r_2=(1-\sqrt{5})/2$
- Compute the general solution $f_n=\alpha_1(r_1)^n+\alpha_2(r_2)^n$
- Find α_1 and α_2 based on the initial conditions:

  $\alpha_1=1/\sqrt{5}$

  $\alpha_2=-1/\sqrt{5}$
- Solution: $f_n=1/\sqrt{5}\cdot((1+\sqrt{5})/2)^n-1/\sqrt{5}\cdot((1-\sqrt{5})/2)^n$