Math/CSE 1019C: Discrete Mathematics for Computer Science
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Jessie Zhao
jessie@cse.yorku.ca

Course page:
http://www.cse.yorku.ca/course/1019

- Test 3
  - Nov 26th, 7pm-8:20pm
  - Ch 2.6, 3.1-3.3, 5.1, 5.2, 5.5
  - 6 questions
  - Locations
    - SLH F (Last name A-L)
    - SLH A (Last name from M-Z)

Recursive Definitions

- Recursive definition: define an object by itself
- Geometric progression $a_n = ar^n$ for n=0,1,2,...
  - A recursive definition: $a_0 = a$, $a_n = r\,a_{n-1}$ for n=1,2,...
- Arithmetic progression $a_n = a+dn$ for n=0,1,2..
  - A recursive definition: $a_0 = a$, $a_n = a_{n-1}+d$ for n=1,2,...

Recursive Definition

- 1. Basis step
  - For functions: specify the value of the function at zero
  - For sets: specify members of the initial set
- 2.Inductive or recursive step
  - For functions: Show how to compute its value at an integer from its values at smaller integers.
  - For sets: Show how to build new things from old things with some construction rules

Practice

- Suppose f(0)=3, and f(n+1)=2f(n)+2, $\forall n \geq 0$. Find f(1), f(2) and f(3).
- Give a recursive definition for f(n)=n!
- Give a recursive definition for f(n)=2

Fibonacci Sequence

- Recursive definition:
  - 1. Basis:
    - f(0)=0, f(1)=1 (two initial conditions)
  - 2.Induction:
    - f(n)=f(n-1)+f(n-2) for n=2,3,4... (recurrence equation)
- Practice: find the Fibonacci numbers f2,f3,f4,f5, and f6

Recursive Definition & Mathematical Induction

- Proof of assertions about recursively defined objects usually involves a proof by induction.
  - Prove the assertion is true for the basis step
  - Prove if the assertion is true for the previous objects it must be true for the new objects you can build from the previous objects
  - Conclude the assertion must be true for all objects

- Example: For Fibonacci numbers prove that $f_n > \alpha^{n-2}$ when $n \geq 3$, where $\alpha = (1+\sqrt{5})/2$
- Proof by strong induction: P(n) is $f_n > \alpha^{n-2}$
  - Basis step:
    - $n=3$, $\alpha < 2 = f_3$, so P(3) is true.
    - $n=4$, $\alpha^2 = ((1+\sqrt{5})/2)^2 = (3+\sqrt{5})/2 < 3 = f_4$, so P(4) is true.
  - Inductive Step:
    - Assume P(j) is true, i.e. $f_j > \alpha^{j-2}$ for $3 \leq j \leq k$, where $k \geq 4$
    - Prove P(k+1) is true, i.e. $f_{k+1} > \alpha^{k-1}$
      - $f_{k+1} = f_k + f_{k-1} \leq \alpha^{k-2} + \alpha^{k-3} = (\alpha+1)\,\alpha^{k-3} = \alpha^2\,\alpha^{k-3} = \alpha^{k-1}$

$\alpha + 1 = \alpha^2$

Practice

- Give a recursive definition for the following sets:
  - Z^+
  - The set of odd positive numbers
  - The set of positive numbers not divisible by 3

Strings

- A string over an alphabet Σ is a finite sequence of symbols from Σ
- The set of all strings (including the empty string λ) is called Σ*
- Recursive definition of Σ*:
  - 1. Basis step: λ∈Σ*
  - 2.Recursive step: If w∈Σ* and x∈Σ, the wx∈Σ*
- Example: If Σ={0,1}, then Σ*={λ,0,1,00,01,10,11,...} is the set of bit strings

- Example: Give a recursive definition of the set S of bit strings with no more than a single 1.
  - 1. Basis step: λ, 0, and 1 are in S
  - 2.Recursive step: if w is in S, then so are 0w and w0.

Length of a String

- Recursive definition of the length of a string l(w)
  - 1. Basis step: l(λ)=0
  - 2.Inductive step: l(wx)=l(w)+1 if w∈Σ* and x∈Σ

Recursive Algorithms

- Input: n: nonnegative integer
- Output: The nth Fibonacci number f(n)

- Fib_recursive(n)
  - If n=0 then return 0
  - Else if n=1 then return 1
  - Else return Fib_recursive(n−1)+Fib_recursive(n−2)

- Correctness proof usually involves strong induction.

Recursion and Iteration

- Iteration: Start with the bases, and apply the recursive definition.

- Fib_iterative(n)
  - If n=0 then return 0
  - Else x←0,y ←1
    - For i ←1 to n−1
      - z ← x+y
      - x ←y
      - y ←z
    - return y

Recursion and Iteration

- Time complexity
  - Fib_recursive(n): exponential
  - Fib_iterative(n): linear

- Drawback of recursion
  - Repeated computation of the same terms

- Advantage of recursion
  - Easy to implement

Recursive Algorithms

- Merge_sort(A,lo,hi)
  - Input: A[1..n]: array of distinct numbers, 1≤lo<hi≤n
  - Output: A[1..n]: A[lo..hi] is sorted

```
If lo=hi
   return;
else
   Mid=⌊(hi+lo)/2⌋
   Merge_sort(A,lo,mid)
   Merge_sort(A,mid+1,hi)
   A[lo,hi]←merge(A[lo,mid],A[mid+1,hi])
```


Correctness proof

- Inductively assume the two recursive calls correct
- Prove the correctness of the merge step
- By strong induction the algorithm is correct

The Basics of Counting

- Why count?
  - arrange objects

- Counting Principles
  - Let A and B be disjoint sets
  - Sum Rule

    $|A \cup B| = |A| + |B|$
  - Product Rule

    $|A \times B| = |A| \cdot |B|$

- Example: Suppose there are 30 men and 20 women in a class.
  - How many ways are there to pick one representative from the class?
    50
  - How many ways are there to pick two representatives, so that one is man and one is woman?
    600

- Example: Suppose you are either going to an Italian restaurant that serves 15 entrees or to a French restaurant that serves 10 entrees
  - How many choices of entree do you have?
    25
- Example: Suppose you go to the French restaurant and find out that the prix fixe menu is three courses, with a choice of 4 appetizers, 10 entrees and 5 desserts
  - How many different meals can you have?
    200

- How many different bit strings of length 8 are there?
  2^8
- How many different bit strings of length 8 both begin and end with a 1?
  2^6
- How many bit strings are there of length 8 or less?
  $2^8 + 2^7 + \ldots + 2^0$ (counting empty string)
- How many bit strings with length not exceeding n, where n is a positive integer, consist no 0.
  $n+1$ (counting empty string)

- How many functions are there from a set with m elements to a set with n elements?
  n^m
- How many one-to-one functions are there from a set with m elements to one with n elements?
  $n(n-1)\ldots(n-m+1)$ when $m \le n$
  0 when $m > n$

Principle of Inclusion-Exclusion

- IF A and B are not disjoint sets
  $|A \cup B| = |A| + |B| - |A \cap B|$
- Don't count objects in the intersection of two sets more than once.
- What is $|A \cup B \cup C|$?

- How many positive integers less than 1000
  - are divisible by 7?
    $\lfloor 999/7 \rfloor = 142$
  - are divisible by both 7 and 11?
    $\lfloor 999/77 \rfloor = 12$
  - are divisible by 7 but not by 11?
    $142 - 12 = 130$
  - are divisible by either 7 or 11?
    $142 + \lfloor 999/11 \rfloor - 12 = 220$
  - are divisible by exactly one of 7 and 11?
    $142 + \lfloor 999/11 \rfloor - 12 - 12 = 208$
  - are divisible by neither 7 nor 11?
    $999 - 220 = 779$