CISC454B: Computer Graphics

- computer graphics is concerned with producing pictures with a computer
  - this is a very broad definition…
- we are interested in three-dimensional computer graphics
  - topics:
    - rendering pipeline
    - mathematical foundations
    - representation of 3D objects
    - camera analogy
    - human vision (brief) and color
    - lighting, material properties, shading
    - rasterization
    - particle systems
    - other topics

Administrative Information

- instructor: Burton Ma
- office hours: 2:30-3:30pm Mon-Thurs Goodwin 735
- course web site
  - www.cs.queensu.ca/home/mab/454.html
  - still under construction
- text books (not required)
  - Computer Graphics Using OpenGL, 2nd Edition
  - OpenGL Programming Guide, 3rd Edition
  - The C++ Programming Language, Special Edition
- class notes
  - available online in PDF format
  - generally not complete (examples will be missing)

Administrative Information (cont)

- programming facilities
  - Walter Light Hall CASLAB
  - 24 Sun Ultra 10 workstations
  - C++, OpenGL

- marking
  - do whatever you want for 100 marks (must write first midterm)

✦ assignments	8 x 6%
✦ midterms	2 x 15%
✦ write a lecture	1 x 10%
✦ final exam	no more than 50%

Administrative Information (cont)

- assignments
  - written and programming
  - can work in groups (up to 3 people per group)
  - no extensions for any reason
    - don't leave them to the last minute
- midterms
  - in class
    - Thursday, February 1
    - Thursday, March 15
  - must write first midterm
  - closed book

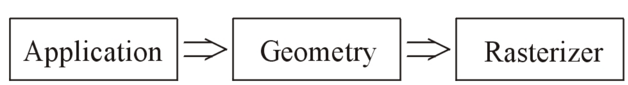
Administrative Information (cont)

- write a lecture
  - produce a lecture based on a research paper
  - can work with one other person
  - due before last week of class
- exam
  - notes and textbook permitted

- comments
  - don't let math intimidate you
  - don't let C++ intimidate you
  - a lot of work
    - budget your time wisely

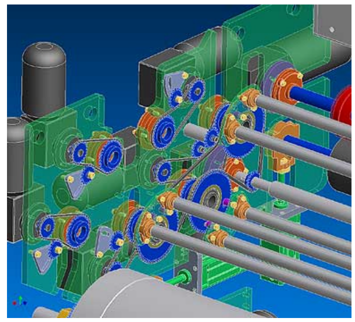
Rendering Pipeline

- rendering modeled as pipeline process
- primitives go in one end, move from stage to stage, and come out at the other end
- pipeline speed determined by slowest stage
- each stage also a pipeline or parallel pipelines
- each stage may discard primitives for efficiency



Application Stage

- application software e.g. video game, computer assisted design (CAD)
  - any application program that needs to send output to the screen



Application Stage (cont)

- defines:
  - geometry to draw (points, lines, polygons, and others)
  - material properties
  - lighting
  - viewing or camera parameters
- also performs other tasks:
  - user interaction (Hill 1.5 for examples of input devices)
  - animation
  - collision detection
  - speed-up techniques
  - many others
- output is scene to be drawn

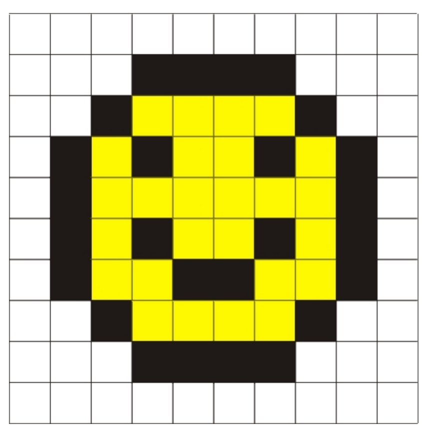
Geometry Stage

- performs most per-polygon and per-vertex operations
- implemented in software or hardware
- Hill calls this stage the graphics pipeline (Figures 5.52, 8.18)

- output is transformed geometry, colour and texture information

Rasterizer Stage

- rasterizes geometry
  - fills in pixels with correct colour to produce final image
- raster image is an array of picture elements (pixels)
  - example:
  - also see Hill 1.3-1.4



Rasterizer Stage (cont)

- implemented in hardware
- performs:
  - hidden surface removal
  - texturing
  - compositing
  - stenciling
  - accumulation
- output is image on screen

Summary

- application stage
  - what to draw and how to draw it
- geometry stage
  - computes 3D appearance of scene from viewer/camera point of view
- rasterizer stage
  - draws 2D screen image

Mathematics for Computer Graphics

- in this course we rely mostly on simple linear algebra
  - more advanced graphics techniques also rely on calculus, statistics, numerical methods
- most of polygon-based computer graphics uses vectors and points defined in 3-dimensional real Cartesian space
- most common family of transformations represented by 4x4 matrix

Vectors

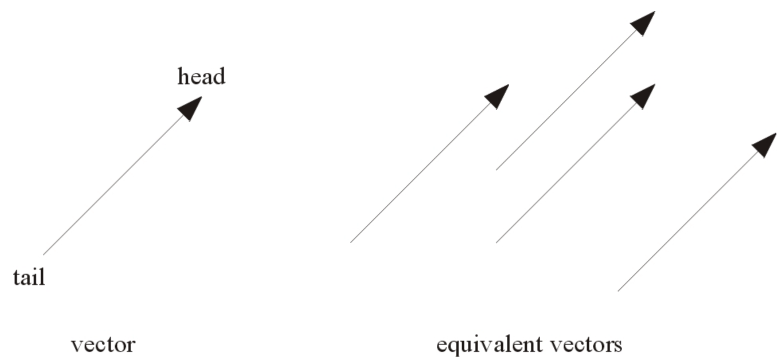
- R^3 is the 3-dimensional real Euclidean space
- vector in R^3 is a 3-tuple of real numbers

$$\vec{v} = (v_0, v_1, v_2) = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_0 & v_1 & v_2 \end{bmatrix}^T$$

$$\vec{w} = (1, 1, -1) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T$$

Drawing Vectors

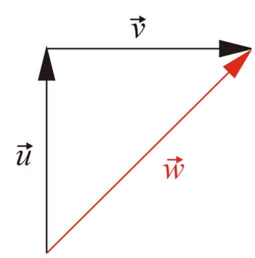
- vector has magnitude and direction
  - but no location



Vector Operations

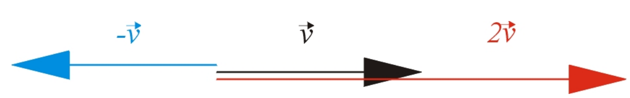
- formally only 2 operations
- vector-vector addition

$$\vec{w} = \vec{u} + \vec{v} = \begin{bmatrix} u_0 + v_0 \\ u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$



- scalar-vector multiplication

$$a\vec{v} = \begin{bmatrix} av_0 \\ av_1 \\ av_2 \end{bmatrix}$$



Vector Properties

$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$	associative
$\vec{u}+\vec{v}=\vec{v}+\vec{u}$	commutative
$\vec{0}+\vec{v}=\vec{v}$	zero identity
$\vec{v}+(-\vec{v})=\vec{0}$	additive inverse
$(ab)\vec{u}=a(b\vec{u})$	associative
$(a+b)\vec{u}=a\vec{u}+b\vec{u}$	distributive
$a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v}$	distributive
$1\vec{u}=\vec{u}$	multiplicative identity

Dot Product

- Hill 4.3
- in Euclidean space dot product (inner product) is defined

$$\begin{aligned} d &= \vec{a}\cdot\vec{b} \\ &= \begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix}\cdot\begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \\ &= a_0b_0 + a_1b_1 + a_2b_2 \end{aligned}$$

- properties

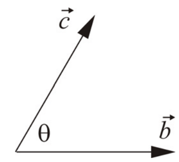
$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$	symmetry
$(\vec{a}+\vec{c})\cdot\vec{b}=\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{b}$	linearity
$(s\vec{a})\cdot\vec{b}=s(\vec{a}\cdot\vec{b})$	homogeneity
$\lvert\vec{b}\rvert^2=\vec{b}\cdot\vec{b}$	magnitude

Dot Product (cont)

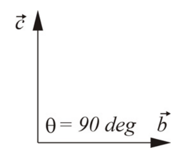
- angle between two vectors (Hill 4.3.2)

  $\vec{b} \cdot \vec{c} = |\vec{b}||\vec{c}|\cos(\vartheta)$

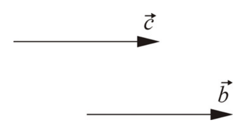
  $\therefore \cos(\vartheta) = \dfrac{\vec{b} \cdot \vec{c}}{|\vec{b}||\vec{c}|}$



- two vectors are perpendicular (orthogonal) if $\vec{b} \cdot \vec{c} = 0$



- two vectors are parallel if $\dfrac{\vec{b} \cdot \vec{c}}{|\vec{b}||\vec{c}|} = 1$



- what can you say about two vectors if their dot product is negative?

Vector Norm

- norm or magnitude of vector defined as

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{{a_0}^2 + {a_1}^2 + {a_2}^2}$$

  - example

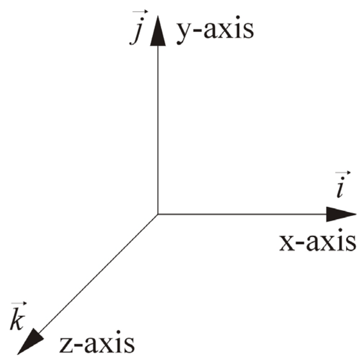
$$\left|[3 \quad 0 \quad -4]^T\right| = \sqrt{3^2 + 0^2 + (-4)^2} = 5$$

- in Euclidean space gives us notion of length or distance
- a unit vector has norm of 1
  - important!
  - to normalize a vector divide by its norm
  - example: normalize $[3\ \ 0\ \ -4]^T$

$$\frac{[3 \quad 0 \quad -4]^T}{5} = [0.6 \quad 0 \quad -0.8]^T$$

Basis

- in R^3 a basis is a set of 3 non-parallel vectors
- common to use orthonormal basis
  - basis vectors are mutually orthogonal
  - basis vectors have unit magnitude
- basis most students are familiar with



$$\vec{i} = [1 \quad 0 \quad 0]^T$$
$$\vec{j} = [0 \quad 1 \quad 0]^T$$
$$\vec{k} = [0 \quad 0 \quad 1]^T$$
$$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0$$
$$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$$

Direction (cont)

- can write any vector as a linear combination of basis vectors

$$\begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix} = -3\vec{i} + 2\vec{j} + 7\vec{k}$$

Cross Product

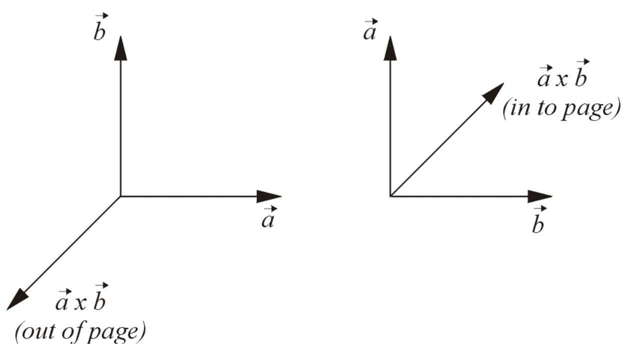
- Hill 4.4
- only defined in R^3
  - defined in terms of standard basis

$$\vec{a}\times\vec{b} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T \times \begin{bmatrix} b_x & b_y & b_z \end{bmatrix}^T$$

$$\begin{aligned} &(a_y b_z - a_z b_y)\vec{i} + \\ =\ &(a_z b_x - a_x b_z)\vec{j} + \\ &(a_x b_y - a_y b_x)\vec{k} \\ =\ &\vec{c} \end{aligned}$$

Cross Product (cont)

- cross product of two vectors is a vector that is orthogonal to the original two vectors
  - direction given by right hand rule



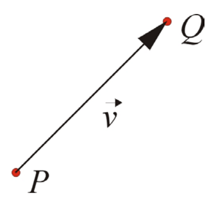
Cross Product (cont)

- properties

$$\begin{aligned} \vec{a}\times\vec{b} &= -\vec{b}\times\vec{a} && \text{antisymmetry} \\ \vec{a}\times(\vec{b}+\vec{c}) &= \vec{a}\times\vec{b}+\vec{a}\times\vec{c} && \text{linearity} \\ (s\vec{a})\times\vec{b} &= s(\vec{a}\times\vec{b}) && \text{homogeneity} \\ \vec{i}\times\vec{j} &= \vec{k} \\ \vec{j}\times\vec{k} &= \vec{i} \\ \vec{k}\times\vec{i} &= \vec{j} \end{aligned}$$

Points

- a point represents location (has zero size)
- to move between points use a vector



$$\begin{aligned} Q &= P+\vec{v} \\ &= \vec{v}+P \end{aligned}$$

- 1 operation defined with points
  - point-point subtraction (yields a vector)

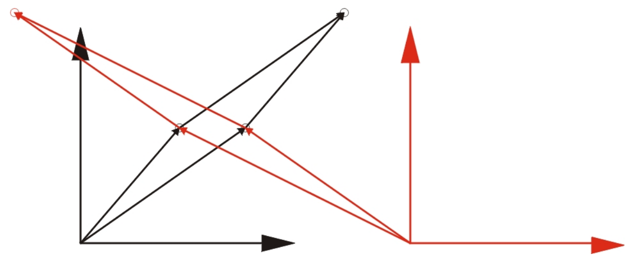
    $Q - P = \vec{v}$

- in Euclidean space distance between two points defined as

  $$\text{distance}(P,Q) = \sqrt{(P-Q)\cdot(P-Q)}$$

Why Only One Operation?

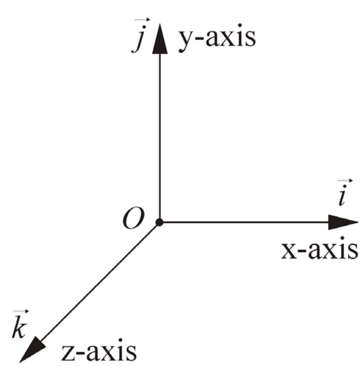
- why can we not add points?
  - not independent of coordinate frame



  - cannot multiply points by scalar for same reason
- but affine sum of points is legal (Hill 4.5.2)

Frames

- very important in computer graphics
  - you've probably been using them since high school
- a frame is a basis and a point called the origin
- most students should be familiar with the standard basis in Cartesian space



$$O = [0 \quad 0 \quad 0]^T$$
$$\vec{i} = [1 \quad 0 \quad 0]^T$$
$$\vec{j} = [0 \quad 1 \quad 0]^T$$
$$\vec{k} = [0 \quad 0 \quad 1]^T$$
$$\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0$$
$$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$$

Homogeneous Representation

- points and vectors are different objects but they look the same

$$P = [x \quad y \quad z]^T \qquad \vec{v} = [x \quad y \quad z]^T$$

- homogeneous representation of points and vectors distinguishes between points and vectors

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

Homogeneous Representation (cont)

- the difference becomes clear when we consider the frame

$$P = [\vec{i} \quad \vec{j} \quad \vec{k} \quad O] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = x\vec{i} + y\vec{j} + z\vec{k} + O$$

$$\vec{v} = [\vec{i} \quad \vec{j} \quad \vec{k} \quad O] \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = x\vec{i} + y\vec{j} + z\vec{k}$$

- notice that
  - vector = linear combination of vectors
  - point = vector + point

Homogeneous Representation (cont)

- to go from ordinary to homogeneous coordinates
  - if the object is a point, append a 1
  - if the object is a vector, append a 0
- to go from homogeneous to ordinary coordinates
  - if the object is a point, delete the 1
    - this rule will change later on
  - if the object is a vector, delete the 0

Points in OpenGL

- OpenGL represents a point with a set of floating-point numbers called a vertex
- to draw a group of points use

```
GLfloat x0, y0, z0, x1, y1, z1, xn, yn, zn;
// assign values to x0, y0, z0, etc. here
// ...
glBegin(GL_POINTS);
glVertex3f(x0, y0, z0);      // point with coordinates (x0, y0, z0)
glVertex3f(x1, y1, z1);
// and so on…
glVertex3f(xn, yn, zn);
glEnd();
```

- every call to glVertex() sends a vertex down the geometry stage

Points in OpenGL (cont)

- many versions of glVertex()

  ```
  void glVertex3f(…)
  ```

- number (here 3) indicates number of coordinates
  - can be 2, 3, or 4
- letter (here f) indicates data type
  - can be
    - s   GLshort
    - i   GLint
    - f   GLfloat
    - d   GLdouble
- examples:

  ```
  glVertex2i(3, 4);
  glVertex3f(-1.0f, 2.0f, 3.5f);
  glVertex4d(1.2, 4.5, 3.9, 1.0);
  ```

Points in OpenGL (cont)

- glVertex() can also take an array as an argument
  - add a “v” to the function name
  - example:

    ```
    GLint one_pt[3] = { 1, 2, 3 };
    GLdouble two_pts[6];
    two_pts[0] = 1.0; two_pts[1] = 2.0; two_pts[2] = 3.0;
    two_pts[3] = 3.0; two_pts[4] = 2.0; two_pts[5] = 1.0;
    glBegin(GL_POINTS);
    glVertex3iv(one_pt);
    glVertex3dv(two_pts);          // point (1.0, 2.0, 3.0)
    glVertex3dv(two_pts+3);        // point (3.0, 2.0, 1.0)
    glEnd();
    ```

Matrices

- only need 3x3 and 4x4 matrices

$$M = \begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix} \qquad M = \begin{bmatrix} m_{00} & m_{00} & m_{00} & m_{00} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix}$$

- identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Vector Multiplication

- can postmultiply a matrix with a column vector

$$\begin{bmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} m_{00}x + m_{01}y + m_{02}z \\ m_{10}x + m_{11}y + m_{12}z \\ m_{20}x + m_{21}y + m_{22}z \end{bmatrix}$$

Matrix Multiplication

- can multiply two 3x3 or two 4x4 matrices together
  - just treat second matrix like 3 or 4 vectors

Matrix Multiplication Properties

$$(LM)N = L(MN)$$
$$L(M + N) = LM + LN$$
$$(L + M)N = LN + MN$$
$$A(sB) = sAB$$
$$MI = IM = M$$
$$MN \neq NM$$

Transpose

- swap rows and columns
  - the transpose of M is M^T

$$M = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix}$$

Transpose Properties

$$\begin{aligned} (aM)^T &= aM^T \\ (M+N)^T &= M^T + N^T \\ (M^T)^T &= M \\ (MN)^T &= N^T M^T \end{aligned}$$

Determinant

- determinant of a matrix is a scalar value
- usually only need 2x2 and 3x3 matrix determinants
  - the determinant of M is $|M|$

$$
\begin{aligned}
|M| &= \begin{vmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{vmatrix} \\
&= m_{00}\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} - m_{01}\begin{vmatrix} m_{10} & m_{12} \\ m_{20} & m_{22} \end{vmatrix} + m_{02}\begin{vmatrix} m_{10} & m_{11} \\ m_{20} & m_{21} \end{vmatrix} \\
&= m_{00}m_{11}m_{22} + m_{01}m_{12}m_{20} + m_{02}m_{10}m_{21} - \\
&\quad - m_{02}m_{11}m_{20} - m_{01}m_{10}m_{22} - m_{00}m_{12}m_{21}
\end{aligned}
$$

Determinant Properties

- for an n x n matrix

$$
\begin{aligned}
|M^{-1}| &= 1/|M| \\
|MN| &= |M||N| \\
|sM| &= s^n|M| \\
|M^T| &= |M|
\end{aligned}
$$

Inverse

- exists only if determinant is nonzero
- multiplicative inverse

$$MM^{-1} = M^{-1}M = I$$

- properties

$$(MN)^{-1} = N^{-1}M^{-1}$$
$$(M^T)^{-1} = (M^{-1})^T$$

- computing inverse?
  - Cramer's rule (we'll see this soon)
  - Gaussian elimination and other methods

Cofactor

- need this for Cramer's rule
- cofactor of matrix element m_{ij} is $(-1)^{i+j}$ times determinant of the matrix obtained by deleting row i and column j from M
  - example (adapted from Hill A2.1.5)

$$M = \begin{bmatrix} 2 & 0 & 6 & 0 \\ 8 & 1 & -4 & 0 \\ 0 & 5 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Adjoint

- adjoint is the transpose of matrix of cofactors

$$\text{cofactor}(M) = C = \begin{bmatrix} 27 & -56 & 40 & 0 \\ 30 & 14 & -10 & 0 \\ -6 & 56 & 2 & 0 \\ 0 & 0 & 0 & 294 \end{bmatrix}$$

$$\text{adjoint}(M) = C^T = \begin{bmatrix} 27 & 30 & -6 & 0 \\ -56 & 14 & 56 & 0 \\ 40 & -10 & 2 & 0 \\ 0 & 0 & 0 & 294 \end{bmatrix}$$

Cramer's Rule

- inverse of M is

$$M^{-1} = \frac{\text{adjoint}(M)}{|M|}$$

Summary

- in graphics the most commonly used concepts are
  - 2x2, 3x3, and 4x4 matrices
  - matrix-vector and matrix-matrix multiplication
  - matrix inverse
- Hill reviews these concepts (and many more) in Appendix 2

Transformations

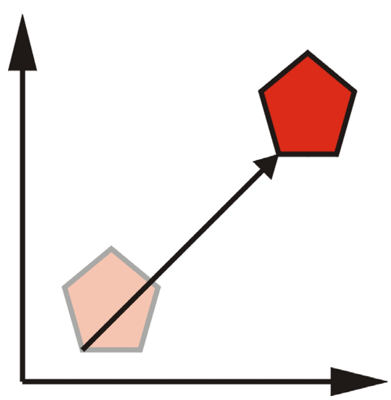
- in graphics, transformations map vectors to vectors and points to points
- transformations can be arbitrarily complex but
  - for efficiency (implementation in geometry pipeline hardware) need to restrict generality of transformations
- we will study the family of affine transformations

Affine Transformations

- transformation T is said to be affine
  - T maps vectors to vectors and points to points
  - T is a linear transformation on vectors
    - $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$
  - $T(P + \vec{v}) = T(P) + T(\vec{v})$
- Hill proves several properties of affine transformations (Section 5.2.7)
- only a few affine transformations
  - translation
  - scale
  - rotation
  - shear
- all can be represented by a 4x4 matrix

Translation

- moves points by a vector amount
- does not affect vectors (because vectors have no location)



Applying Translation

- translation leaves vectors unchanged

$$\vec{v} = \begin{bmatrix} x & y & z & 0 \end{bmatrix}^T$$

$$T\vec{v} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

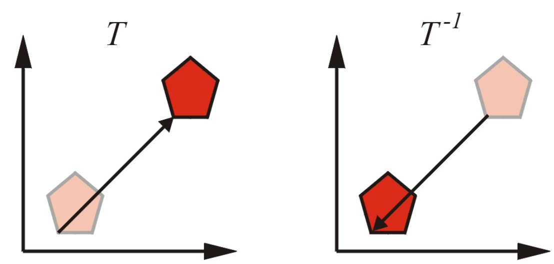
- translation moves points by a vector amount

$$P = \begin{bmatrix} x & y & z & 1 \end{bmatrix}^T$$

$$TP = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$

Inverse of Translation

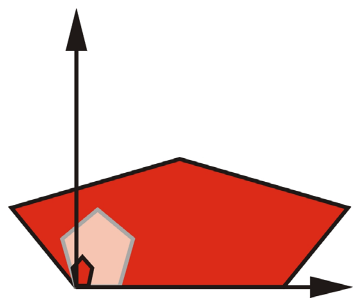
- inverse of a transformation undoes the transformation



- check that $TT^{-1} = T^{-1}T = I$

Scale

- enlarge or shrink an object
- scales objects about the x, y, and z-directions
  - origin is invariant



- if $0 < sx < 1$, then object shrinks by a factor of sx in x-direction
- if $sx > 1$, then object grows by a factor of sx in x-direction
- what if $sx < 0$? if $sx = 0$?

Inverse of Scale

- if an object is scaled by a factor of s
  - then the inverse scales by a factor of 1/s

Shear

- six basic shearing transformations

$$H_{xy}=\begin{bmatrix}1&h&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad H_{xz}=\begin{bmatrix}1&0&h&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad H_{yx}=\begin{bmatrix}1&0&0&0\\h&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

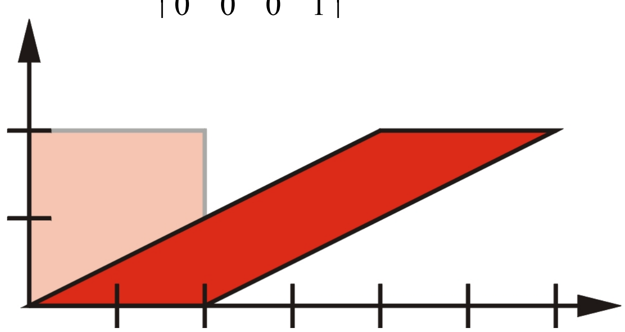
$$H_{yz}=\begin{bmatrix}1&0&0&0\\0&1&h&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad H_{zx}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\h&0&1&0\\0&0&0&1\end{bmatrix}\quad H_{zy}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&h&1&0\\0&0&0&1\end{bmatrix}$$

- first subscript: which coordinate is changed
- second subscript: which coordinate does the shearing

Shear Example

- suppose we have h=2 for Hxy

$$H_{xy}=\begin{bmatrix}1&2&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$



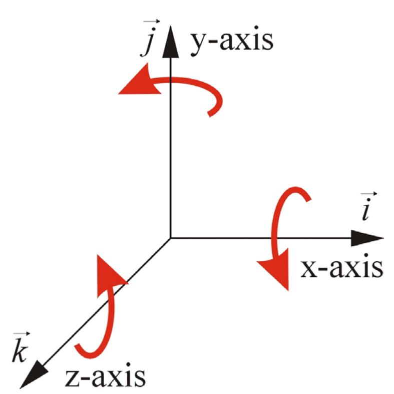
- invariant points?

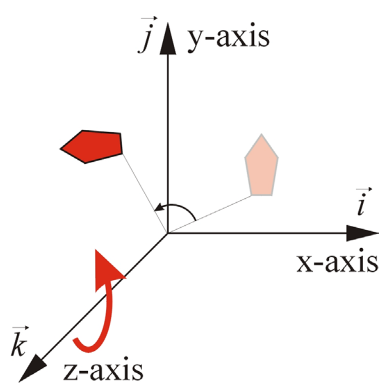
Inverse of Shear

- think about it

Rotation

- rotation about x, y, and z-axis
- points on axis of rotation are invariant
- positive angle of rotation causes a counterclockwise rotation about the axis when you look along the axis towards the origin





Rotation (cont)

- three basic rotation matrices (one for each axis)
  - check that points on the axes of rotation are invariant

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) & 0 \\ 0 & \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_y = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 & 0 \\ \sin(\beta) & \cos(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse of Rotation

- rotation matrix is orthogonal
- fact: inverse of an orthogonal matrix is the transpose
  - for ANY rotation matrix: $R^{-1} = R^T$
- geometrically
  - if you rotate about an axis by β degrees then the inverse is a rotation about the same axis by -β degrees

Composition or Concatenation of Transformations

- rare to perform only one elementary transformation
- composition of affine transformations is also affine
- order transformations are applied in matters
  - matrix multiplication does not commute
  - example: translate then scale vs. scale then translate

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Composition or Concatenation of Transformations

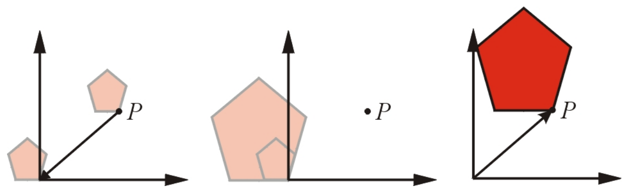
- reading left to right transformation matrices appear in reverse order
  - example: apply A then B then C

$$A\vec{a} = \vec{b}$$
$$B\vec{b} = \vec{c}$$
$$C\vec{c} = \vec{d}$$
$$\therefore \vec{d} = C(B(A\vec{a}))$$

  - overall transformation is $T = CBA$

Composition or Concatenation of Transformations (cont)

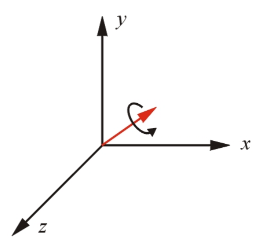
- example: scale about arbitrary point P = (px, py, pz)
  - translate P to origin
  - scale
  - translate back to original point P



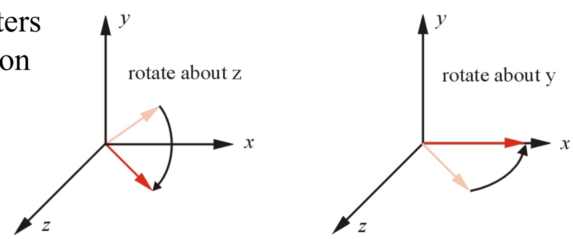
$$\begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -p_x \\ 0 & 1 & 0 & -p_y \\ 0 & 0 & 1 & -p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Composition or Concatenation of Transformations (cont)

- example: rotation about arbitrary axis (hard way)
  - apply two rotations to align axis with x-axis
    - illustrated right and below
  - rotate about x-axis
  - undo first two rotations

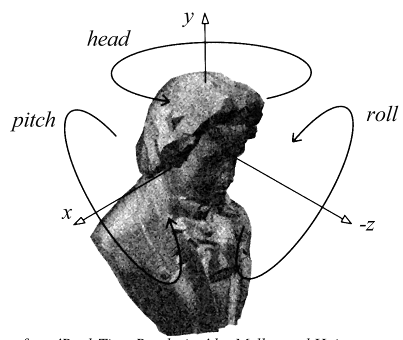


only need 3 parameters
to specify a rotation



Rotations Revisited

- rotations are common
- many different ways of specifying arbitrary rotation
- example: Euler transformations
  - 24 different Euler transformations
  - head (yaw), pitch, roll is common



from 'Real-Time Rendering' by Moller and Haines

$$R(h,p,r) = R_z(r)R_x(p)R_y(h)$$

Rotations Revisited (cont)

- Goldman (in Graphics Gems 1)
- for rotation of β degrees about an axis with normalized direction vector $\vec{u} = (u_x, u_y, u_z)$

$c =$

$s =$

$t =$

$$R = \begin{bmatrix} c + tu_x^2 & tu_xu_y - su_z & tu_xu_z + su_y & 0 \\ tu_xu_y + su_z & c + tu_y^2 & tu_yu_z - su_x & 0 \\ tu_xu_z - su_y & tu_yu_z - su_x & c + tu_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- don’t bother memorizing this

Affine Transformations and the Determinant

- the determinant of an affine transformation matrix tells you how much the transformation scales the volume of an object by
  - if an object D has volume V then applying an affine transformation M to the object produces a new object with volume |M|V
- you only need to compute the determinant of the upper left 3x3 matrix

$$|M| = \begin{vmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -0\begin{vmatrix} m_{01} & m_{02} & m_{03} \\ m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{vmatrix} + 0\begin{vmatrix} m_{00} & m_{02} & m_{03} \\ m_{10} & m_{12} & m_{13} \\ m_{20} & m_{22} & m_{23} \end{vmatrix} - 0\begin{vmatrix} m_{00} & m_{01} & m_{03} \\ m_{10} & m_{11} & m_{13} \\ m_{20} & m_{21} & m_{23} \end{vmatrix} + 1\begin{vmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{vmatrix}$$

Interpreting Transformations

- we have assumed that affine transformations transform points and vectors
- this is not the only interpretation
  - transformation can transform the coordinate frame
    - this is a common interpretation in OpenGL
    - we'll see this a little later
  - transformation can transform from one affine space to another affine space

Affine Transformations in OpenGL

- OpenGL maintains a stack of transformations called modelview matrix stack
- several functions modify the top-of-stack element by postmultiplying top-of-stack with a matrix

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();              // tos = I
glMultMatrixf(N);              // tos = I*N
glMultMatrixf(M);              // tos = I*N*M
glMultMatrixf(L);              // tos = I*N*M*L
glBegin(GL_POINTS);
glVertex3f(x, y, z);           // transformed by tos
glEnd();
```

- transforms vertex by N*M*L
- top-of-stack is called current transformation

Translation, Scale, and Rotation

`void glTranslatef(GLfloat x, GLfloat y, GLfloat z);`

- postmultiplies current transformation by translation matrix T(x,y,z)

`void glScalef(GLfloat x, GLfloat y, GLfloat z);`

- postmultiplies current transformation by scale matrix S(x,y,z)

`void glRotatef(GLfloat angle, GLfloat x, GLfloat y, GLfloat z);`

- postmultiplies current transformation by rotation matrix corresponding to rotation of angle degrees about the axis from the origin to the point (x,y,z)

- OpenGL calls these transformations modeling transformations

Other Affine Transformations

- notice that no shear function
- must specify all 16 values of transformation matrix for “custom” transformations
  - OpenGL requires an array with the 16 elements specified like so:

$$\begin{bmatrix} m_0 & m_4 & m_8 & m_{12} \\ m_1 & m_5 & m_9 & m_{13} \\ m_2 & m_6 & m_{10} & m_{14} \\ m_3 & m_7 & m_{11} & m_{15} \end{bmatrix}$$

```
GLfloat S[16];          // a scale matrix
S[0] = 3.0f;   S[4] = 0.0f;   S[8] = 0.0f;    S[12] = 0.0f;
S[1] = 0.0f;   S[5] = 5.0f;   S[9] = 0.0f;    S[13] = 0.0f;
S[2] = 0.0f;   S[6] = 0.0f;   S[10] = 7.0f;   S[14] = 0.0f;
S[3] = 0.0f;   S[7] = 0.0f;   S[11] = 0.0f;   S[15] = 1.0f;
```


Other Affine Transformations (cont)

```
void glLoadMatrixf(const GLfloat* M);
```

- sets the 16 values of current transformation matrix to those in the array M

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();            // tos = I
glLoadMatrixf(S);            // tos = I*S
```

```
void glMultMatrixf(const GLfloat* M);
```

- postmultiplies current transformation by matrix defined by M
- remember: if current matrix is C then current matrix is replaced with C*M

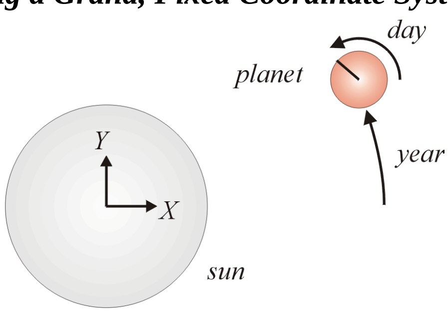
Thinking About Transformations in OpenGL

- a single grand, fixed coordinate system (often called the ‘world’)
  - matrix multiplications affect position, orientation, and size of objects
  - this is how we’ve been interpreting transformations so far
- transformations are specified in opposite order
- example: rotation followed by translation

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
glMultMatrixf( T );        // or glTranslatef()
glMultMatrixf( R );        // or glRotatef()
// draw object here…
```

- example: sun and planet

Using a Grand, Fixed Coordinate System



- assume that we know how to draw a sphere centered at the origin
- the sun is at already at the origin, we just have to draw it
- the planet needs to be transformed to its orientation and position starting from the origin
  - rotate by ‘day’ degrees about z-axis
  - translate in x-direction by radius of planet’s orbit
  - rotate by ‘year’ degrees about z-axis

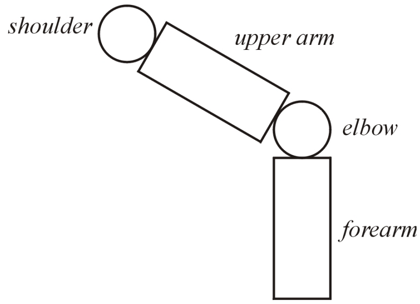
Using a Grand, Fixed Coordinate System (cont)

- in OpenGL

```
glMatrixMode(GL_MODELVIEW);
drawSun();          // draws a sphere at the origin with sun size
glRotatef(year, 0.0f, 0.0f, 1.0f);
glTranslatef(orbit, 0.0f, 0.0f);
glRotatef(day, 0.0f, 0.0f, 1.0f);
drawPlanet();       // draws a sphere at the origin with planet size
```

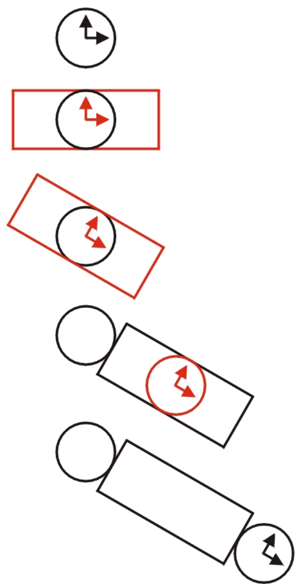

Using a Local Coordinate System

- instead of a world coordinate system, consider a coordinate system local to model
  - matrix multiplications affect position, orientation, and scale of local coordinate frame
- transformations appear in "natural" order
- especially useful for drawing articulated or hierarchical models



position of objects are related to one another

Using a Local Coordinate System (cont)



drawCircle();   // need to define this

glRotatef(-30.0f, 0.0f, 0.0f, 1.0f);

glTranslatef(5.0f, 0.0f, 0.0f);
drawRectangle(); // need to define this

glTranslatef(5.0f, 0.0f, 0.0f);
drawCircle();

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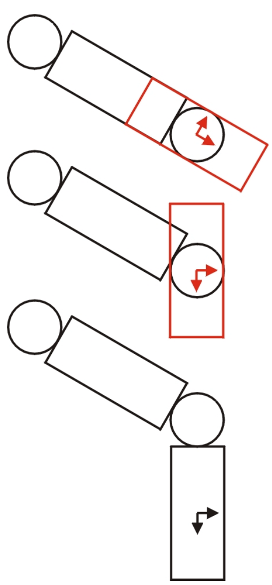
Using a Local Coordinate System (cont)



glRotatef(-60.0f, 0.0f, 0.0f, 1.0f);

glTranslatef(5.0f, 0.0f, 0.0f);
drawRectangle();

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Using a Local Coordinate System (cont)

- beware if you use scale transformations when thinking in terms of a local coordinate system
  - glScalef() will change the scale of the coordinate axes!
- we can apply an inverse scale (after we're done with the original scale) but there is a better way
  - we can manipulate the matrix stack
    - we'll study this a little later on

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Affine Transformations Summary

- affine transformations in 3D can be represented with a 4x4 matrix
- four different types of basic affine transformations and their inverses
  - translation, scaling, shear, rotation
- when applying multiple transformations, write matrices from right to left (if you think of transforming points and vectors)
- remember how to invert a concatenation of transformations
- determinant of an affine transformation matrix tells you the factor by which the volume of an object changes when you apply the transformation to the object

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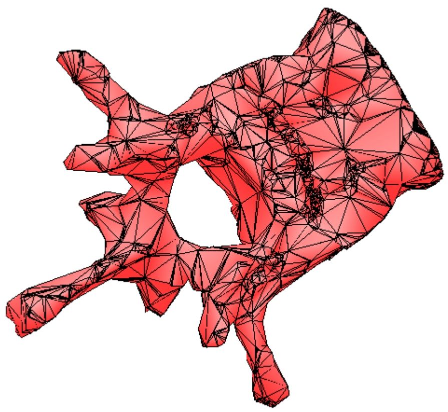
Representation of Object Surfaces

- most common representation of objects is polygonal mesh/net
  - collection of polygons that approximate the outer surface or skin of the object



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Representation of Object Surfaces

- modern hardware capable of rendering simple polygons fast
  - NVIDIA GeForce2 Ultra: 31 million polygons/s
  - PlayStation2: more than 60 million polygons/s (raw speed)
    - many factors can affect these numbers (polygon size, image size, lighting, type of shading, etc) so don’t take them at face value
- if polygons are small enough (i.e. if sufficiently large number of polygons are used) resulting images can be realistic
  - “reality is 80,000,000 polygons per frame”
    - Carpenter, Catmull, and Cook
  - 2.4 billion polygons per second
    - complexity of scenes grows faster than hardware speed
    - we’re still many years away from this number

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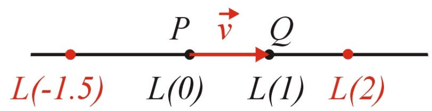
Lines

- a line is 1-dimensional
    - has infinite length, but no other dimension
- a line is defined by 2 noncoincident points P and Q
    - or by a point P and a vector parallel to the line
        - any point L on a line is given by:

$$
\begin{aligned}
L &= L(t) \\
&= P + (Q - P)t \\
&= P + \vec{v}t \\
&= (1-t)P + tQ \quad \text{an affine sum of points}
\end{aligned}
$$



- L(t) is called the parametric form of a line
- can produce a finite line (called a line segment) by restricting the domain of L(t)

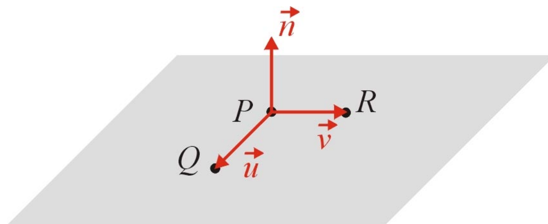
Planes

- a plane is 2-dimensional
    - has infinite length and width, but no other dimension
- a plane can be defined by 3 noncollinear points P, Q, and R in the plane
    - or by a point P and two nonparallel vectors parallel to the plane
        - any point on the plane is given by:

$$
\begin{aligned}
A &= A(s,t) \\
&= P + (Q - P)s + (R - P)t \\
&= P + \vec{u}s + \vec{v}t \\
&= (1-s-t)P + sQ + tR \quad \text{an affine sum of points}
\end{aligned}
$$

- A(s,t) is the parametric form of the plane

Planes (cont)



- a plane can also be defined by a point P and a vector perpendicular to the plane
  - for every point X in the plane

    $$\vec{n} \cdot (X - P) = 0$$

  - this is called the point-normal equation of a plane
- the vector $\vec{n}$ is called the normal vector to the plane
- given P, Q, and R it is easy to compute $\vec{n}$

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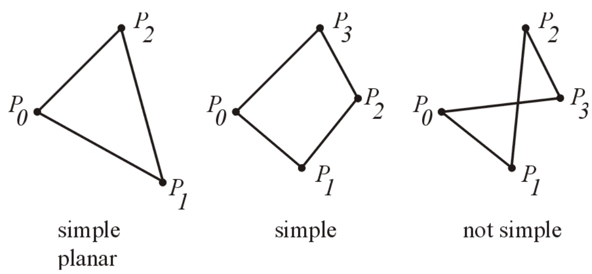
Polygons



- a polygon is an ordered set of points (vertices) with adjacent points connected by edges (line segments)
  - polygons are closed: first and last points are connected
- we will use counterclockwise convention (when looking at the outside surface or front face of the polygon)
- a polygon is simple if no two edges intersect
- a polygon is planar if it is mathematically flat (contained by a plane)

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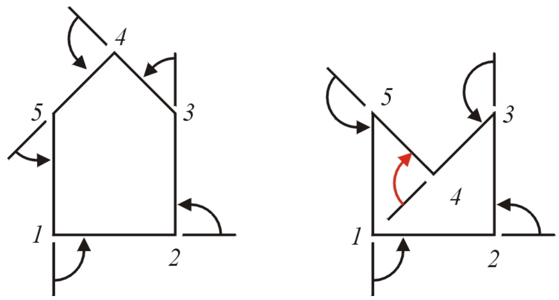
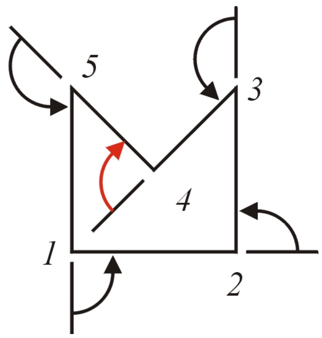
Polygons: Turning Angles



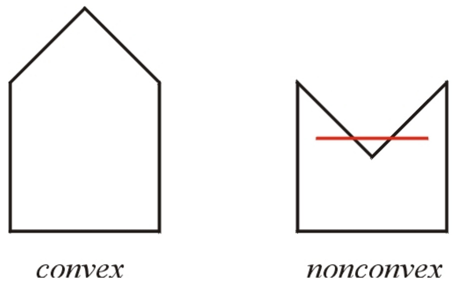
- angle by which you turn at vertex called turning angle
  - for counterclockwise ordering of vertices
    - turn left: turning angle is positive
    - turn right: turning angle is negative
- interior angle = 180 – turning angle (degrees)
  - or π - turning angle (radians)

Polygons: Turning Angles (cont)



- how do you compute the sign of the turning angle?
  - hint: at vertex P_i consider the edge vectors (P_i-P_{i-1}) and (P_{i+1} - P_i)
    - now consider the normal vector of the polygon

Polygons: Convexity



- a polygon is convex if it has no indentations
  - any two points in a convex polygon can be connected with a straight line that never leaves the polygon
  - all interior angles less than 180 degrees (π radians)
  - all turning angles have same sign
- a nonconvex polygon also called concave

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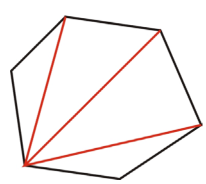
Polygons: Winding Number

- sum of interior angles = (n-2)*180 degrees for convex polygon of n sides
  - proof:



- sum of turning angles = 360 degrees for convex polygon
  - proof?
- winding number = (sum of all turning angles) / (360 degrees)
  - for a convex polygon, the winding number = 1

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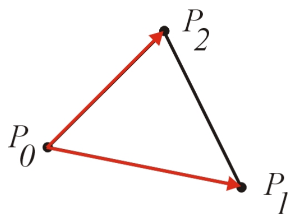
Polygonal Meshes

- many algorithms assume triangular meshes
  - hardware support
  - always convex
  - always planar
    - polygon normal vector easy to compute

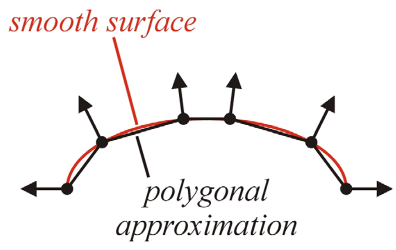


$$\vec{N} = (P_1 - P_0) \times (P_2 - P_0)$$

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|}$$

Polygonal Meshes: Per Vertex Normal Vectors

- a mesh usually approximation for smooth surface
  - for shading want normal vector of smooth surface
    - store this information only at mesh vertices
      - example: normal vectors shown as arrows, vertices as dots
      - important: normal vectors are perpendicular to “true” smooth surface



- usually most convenient to store normalized (unit) normal vectors

Polygonal Meshes: Operations

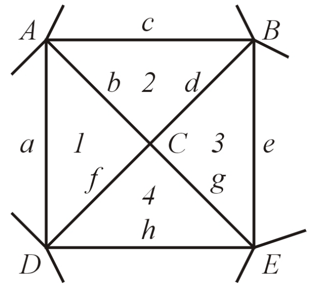
- rendering
- simplification
  - given a mesh, compute a new mesh that looks the same as the old mesh but has fewer vertices and faces
- smoothing
  - given a mesh, compute a new mesh that looks smoother than the old mesh
- animation or warping
- slicing
  - cut a mesh into two or more meshes

Polygonal Meshes: Operations (cont)

- adjacency relationship queries

Given	Find all adjacent
vertex	vertices
	edges
	faces
edge	vertices
	edges
	faces
face	vertices
	edges
	faces

Polygonal Meshes: Operations (cont)



- adjacency examples
- vertex C is adjacent to:
  - vertices A, B, D, E
  - edges b, d, f, g
  - faces 1, 2, 3, 4
- edge b is adjacent to:
  - vertices A, C
  - edges a, c, d, f, g, and others
  - faces 1, 2
- face 2 is adjacent to:
  - vertices A, B, C
  - edges b, c, d
  - faces 1, 3, and one other

Polygonal Meshes: Data Structures

- efficiency
  - memory or storage
  - time to access specific geometry
  - time to perform specific operations (e.g. answer adjacency query)
  - of rendering?
- meshes often store
  - position of vertices (geometry)
  - how the vertices are connected (topology)
  - normal direction at vertices (orientation)
    - other stuff too
      - material properties
      - texture coordinates
      - colors

Polygonal Meshes: A Simple Data Structure

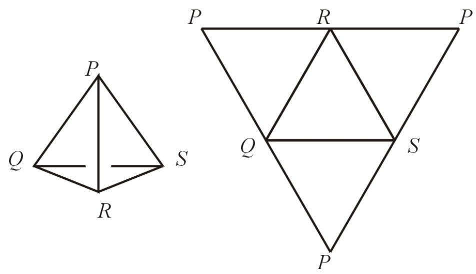
- mesh is a collection of polygons (commonly called faces)
- simplest data structure stores every face
  - example for triangle mesh

face	vertex 0	normal 0	vertex 1	normal 1	vertex 2	normal 2
0	x_{00} y_{00} z_{00}	n_{x00} n_{y00} n_{z00}	x_{01} y_{01} z_{01}	n_{x01} n_{y01} n_{z01}	x_{02} y_{02} z_{02}	n_{x02} n_{y02} n_{z02}
1	x_{10} y_{10} z_{10}	n_{x10} n_{y10} n_{z10}	x_{11} y_{11} z_{11}	n_{x11} n_{y11} n_{z11}	x_{12} y_{12} z_{12}	n_{x12} n_{y12} n_{z12}
etc						

Polygonal Meshes: A Simple Data Structure (cont)

- storage requirements
  - each vertex requires 3 floating point numbers
  - each vertex normal requires 3 floating point numbers
  - each face has 3 vertices and 3 vertex normals
    - F faces require 18*F floating point numbers
- vertices repeated
- normals repeated
- information about edges not explicit
- adjacency operations are inefficient
- rendering straightforward but inefficient

Polygonal Meshes: A Simple Data Structure Example



tetrahedron is not a smooth surface
so normal vectors are constant
for each face

face	vertices	normals
0	P_x, P_y, P_z Q_x, Q_y, Q_z R_x, R_y, R_z	a_x, a_y, a_z a_x, a_y, a_z a_x, a_y, a_z
1	R_x, R_y, R_z Q_x, Q_y, Q_z S_x, S_y, S_z	b_x, b_y, b_z b_x, b_y, b_z b_x, b_y, b_z
2	R_x, R_y, R_z S_x, S_y, S_z P_x, P_y, P_z	c_x, c_y, c_z c_x, c_y, c_z c_x, c_y, c_z
3	Q_x, Q_y, Q_z P_x, P_y, P_z S_x, S_y, S_z	d_x, d_y, d_z d_x, d_y, d_z d_x, d_y, d_z

Polygonal Meshes: Shared Vertex Data Structure

- avoid repetitive storage of vertices
  - store each vertex only once
- requires
  - vertex list to store geometric information
    - store each distinct vertex once
  - normal list to store orientation information
    - store each distinct normal vector once
    - not necessarily equal to number of vertices
  - face list to store connectivity or topological information
    - each face stores pointers or array indices or identifiers into the vertex and normal lists
- this is the mesh format Hill uses

Polygonal Meshes: Shared Vertex Data Structure (cont)

- vertex list (V vertices)

vertex	coordinates
0	x_0, y_0, z_0
1	x_1, y_1, z_1
…	
V-1	$x_{V-1}, y_{V-1}, z_{V-1}$

- normal list (N normal vectors)

normal vector	coordinates
0	nx_0, ny_0, nz_0
1	nx_1, ny_1, nz_1
…	
N-1	$nx_{N-1}, ny_{N-1}, nz_{N-1}$

Polygonal Meshes: Shared Vertex Data Structure (cont)

- face table (F faces)
  - note that numbers in vertices and normals columns are for example only

face	vertices	vertex normals
0	0, 4, 5 (array indices into vertex list)	0, 1, 2 (array indices into normal list)
1	3, 6, 9	5, 1, 9
…		
F-1	the vertices of the face in counterclockwise order	the normal vector associated with each of the vertices

Polygonal Meshes: Shared Vertex Data Structure (cont)

- storage requirements?
  - need relationship between number of vertices and number of faces
- if the mesh has no holes (e.g. not a doughnut or torus), and if every edge is shared by exactly two polygons
  - Euler's formula: V – E + F = 2
    - # vertices – # edges + # faces = 2
  - triangle mesh: 3F ≈ 2E
  - under these assumptions:

$$
\begin{aligned}
V - E + F &= 2 \\
V - 3F/2 + F &\approx 2 \\
V - F/2 &\approx 2 \\
V &\approx F/2
\end{aligned}
$$

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Polygonal Meshes: Shared Vertex Data Structure (cont)

- what about the number of normal vectors?
  - impossible to say in general, but assume N ≈ V
  - the storage requirements are:

$$
\begin{aligned}
&3V + 3N + 6F \\
&\approx 3V + 3V + 6F \\
&= 6V + 6F \\
&\approx 3F + 6F \\
&= 9F
\end{aligned}
$$

- this is half the storage requirement of the first simple data structure

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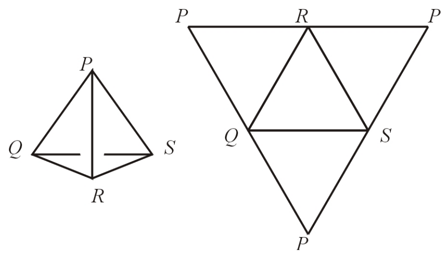
Polygonal Meshes: Shared Vertex Data Structure Example



vertex	coordinates
P	P_x, P_y, P_z
Q	Q_x, Q_y, Q_z
R	R_x, R_y, R_z
S	S_x, S_y, S_z

normal	coordinates
a	a_x, a_y, a_z
b	b_x, b_y, b_z
c	c_x, c_y, c_z
d	d_x, d_y, d_z

face	vertex	normal
0	P, Q, R	a, a, a
1	R, Q, S	b, b, b
2	R, S, P	c, c, c
3	Q, P, S	d, d, d