

# CSE4421/5324: Assignment 1

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1. Find the  $4 \times 4$  homogeneous transformation matrix (showing the numeric values for all 16 elements)  $T_1^0$  where:

- (a)  $\{1\}$  has the same orientation as  $\{0\}$  and the origin of  $\{1\}$  is translated relative to the origin of  $\{0\}$  by  $d_1^0 = [5 \ -5 \ 10]^T$ .
- (b) The origin of  $\{1\}$  is coincident with the origin of  $\{0\}$ , and  $\hat{x}_1^0 = -\hat{z}_0^0$ ,  $\hat{y}_1^0 = -\hat{x}_0^0$ , and  $\hat{z}_1^0 = \hat{y}_0^0$ .
- (c) The origin of  $\{1\}$  is translated relative to the origin of  $\{0\}$  by  $d_1^0 = [0 \ 0 \ -10]^T$ , and the orientation of  $\{1\}$  relative to  $\{0\}$  is the same as in part (b).
- (d) The origin of  $\{0\}$  is translated relative to the origin of  $\{1\}$  by  $d_0^1 = [0 \ 0 \ -10]^T$ , and the orientation of  $\{1\}$  relative to  $\{0\}$  is

$$\begin{aligned}\hat{x}_1^0 &= [0.9971 \ -0.0292 \ -0.0705]^T, \\ \hat{y}_1^0 &= [-0.0292 \ 0.7083 \ -0.7053]^T, \text{ and} \\ \hat{z}_1^0 &= [0.0705 \ 0.7053 \ 0.7053]^T.\end{aligned}$$

Explain how you derived the solution for part (d).

2. Find the missing elements of the following rotation matrices. Show your work, or explain your reasoning. It may be the case that there is no unique solution, in which case you should find all possible solutions.

(a)  $\begin{bmatrix} \cdot & 1 & 0 \\ \cdot & 0 & 0 \\ \cdot & 0 & -1 \end{bmatrix}$

(b)  $\begin{bmatrix} \cdot & \sqrt{3}/2 & 0 \\ \cdot & 0 & 1 \\ \sqrt{3}/2 & \cdot & 0 \end{bmatrix}$

(c)  $\begin{bmatrix} 0 & 0 & 1 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \end{bmatrix}$

3. Consider the following  $4 \times 4$  homogeneous transformation matrices:

- $R_{x,a}$  : rotation about  $x$  by an angle  $a$
- $R_{y,a}$  : rotation about  $y$  by an angle  $a$
- $R_{z,a}$  : rotation about  $z$  by an angle  $a$
- $D_{x,a}$  : translation along  $x$  by a distance  $a$
- $D_{y,a}$  : translation along  $y$  by a distance  $a$
- $D_{z,a}$  : translation along  $z$  by a distance  $a$

Write the matrix product giving the overall transformation for the following sequences (do not perform the actual matrix multiplications):

(a) The following rotations all occur in the moving frame.

- i. Rotate about the current  $x$ -axis by angle  $\phi$ .
- ii. Rotate about the current  $y$ -axis by angle  $\theta$ .
- iii. Rotate about the current  $z$ -axis by angle  $\psi$ .

*Note: This yields the XYZ-Euler angle rotation matrix.*

(b) The following rotations all occur in a fixed (world) frame.

- i. Rotate about the world  $x$ -axis by angle  $\psi$ .
- ii. Rotate about the world  $y$ -axis by angle  $\theta$ .
- iii. Rotate about the world  $z$ -axis by angle  $\phi$ .

*Note: This yields the roll, pitch, yaw (RPY) rotation matrix.*

(c) The desired matrix product for this question is  $T_2^0$ .

Frame  $\{1\}$  is obtained starting from frame  $\{0\}$  using transformations in the moving frame:

- i. Rotate about the current  $z$ -axis by angle  $\theta$ .
- ii. Translate along the current  $x$ -axis by a distance  $d$ .

Frame  $\{2\}$  is obtained starting from frame  $\{1\}$  using transformations in the fixed frame  $\{1\}$ :

- i. Rotate about the  $x$ -axis of  $\{1\}$  by angle  $\phi$ .
- ii. Rotate about the  $y$ -axis of  $\{1\}$  by angle  $\psi$ .
- iii. Translate along the  $z$ -axis of  $\{1\}$  by  $d$ .

4. Consider one link of a robotic arm that is commanded to move (relative to the previous link) by a transformation  $T_j^i$ . The actual motion is measured externally (perhaps using a second robotic arm) and is found to be  $\tilde{T}_j^i$ . Find an expression for the transformation needed to correct the actual motion so that it matches the commanded transformation.

5. Rigid transformations are useful in many applications; for example, in computer-aided surgery rigid transformations are used to represent the relationship between coordinate frames and the desired anatomical correction.

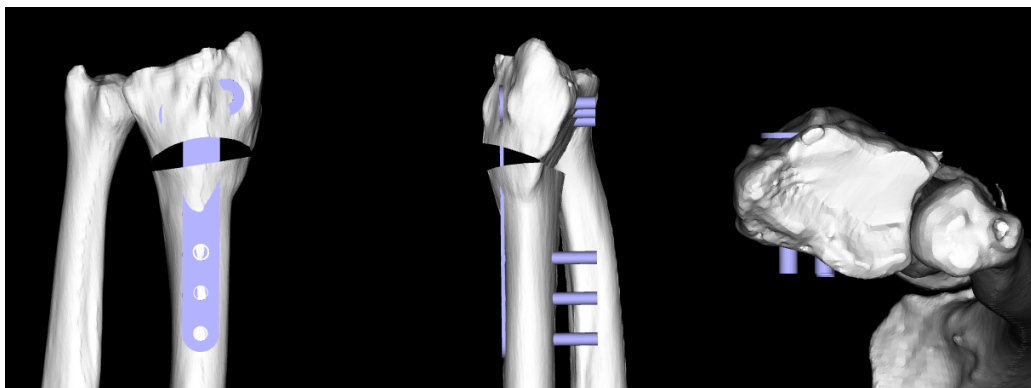


Figure 1: The planned anatomical correction for a distal radial osteotomy (DRO). In this operation, the radius is cut into two fragments and the distal end (the end closest to the wrist) is translated and rotated to restore the bone to its normal anatomical position; this correction transformation can be planned virtually using pre-operative medical images (such as a CT scan). The transformation is defined in frame  $\{0\}$ , the frame associated with the medical images. This operation is used to correct a previously fractured radius that has healed incorrectly.

In the operating room, coordinate frame  $\{1\}$  is associated with the patient. It is possible to establish the pose of  $\{1\}$  with respect to  $\{0\}$ ,  $T_0^1$ , through a process called registration. The problem facing the surgeon is that the surgeon must perform the correction transformation in frame  $\{1\}$ , but the transformation is defined in frame  $\{0\}$ .

Suppose that the correction transformation is given in frame  $\{0\}$  by the  $4 \times 4$  homogeneous matrix  $C$ . What is  $C$  expressed in frame  $\{1\}$ ? Explain how you derived your solution.

## 6. Graduate students only

A quaternion is another representation of rotation in 3D. The quaternion  $Q = (q_w, q_x, q_y, q_z)$  can be thought of as being a scalar  $q_w$  and a vector  $\vec{q} = [q_x \ q_y \ q_z]^T$ . Given two quaternions  $A = (a_w, \vec{a})$  and  $B = (b_w, \vec{b})$ , the quaternion product  $C = AB$  is defined as

$$\begin{aligned} c_w &= a_w b_w - \vec{a} \cdot \vec{b} \\ \vec{c} &= a_w \vec{b} + b_w \vec{a} + \vec{a} \times \vec{b} \end{aligned}$$

where  $\vec{a} \times \vec{b}$  is the cross product of  $\vec{a}$  and  $\vec{b}$ .

- Show that  $Q_I Q = Q Q_I = Q$  for every unit quaternion  $Q$  where  $Q_I = (1, 0, 0, 0)$ , i.e.,  $Q_I$  is the identity quaternion.
- The conjugate  $Q^*$  of a quaternion  $Q = (q_w, \vec{q})$  is given by  $Q^* = (q_w, -\vec{q})$ . Show that  $Q^* Q = Q Q^* = (1, 0, 0, 0)$ , i.e.,  $Q^*$  is the inverse of  $Q$ .
- The quaternion  $Q = (q_w, \vec{q})$  where  $q_w = \cos \frac{\theta}{2}$  and  $\vec{q} = [k_x \sin \frac{\theta}{2} \ k_y \sin \frac{\theta}{2} \ k_z \sin \frac{\theta}{2}]^T$  represents the rotation of angle  $\theta$  about the unit vector  $\hat{k} = [k_x \ k_y \ k_z]^T$ . A vector  $\vec{p} =$

$[p_x \ p_y \ p_z]^T$  can be rotated using the quaternion product  $QPQ^*$  where  $P$  is the quaternion  $(0, \vec{p})$ . Show that this is true for a rotation of angle  $\theta$  about the  $z$ -axis.

### 7. Graduate students only

Consider a vector  $v$  that is rotated about a unit vector  $\hat{k}$  (passing through the origin) by an angle  $\theta$  to form a new vector  $v'$ :

$$v' = R_{\hat{k}, \theta} v$$

Derive Rodrigues' rotation formula,

$$v' = v \cos \theta + (\hat{k} \times v) \sin \theta + \hat{k}(\hat{k} \cdot v)(1 - \cos \theta)$$

Do not replicate the Wikipedia derivation; instead, use the rotation matrix for rotation about an axis  $\hat{k}$  by an angle  $\theta$ .

### 8. Programming Question for everyone

Complete the programming problem from Lab 2 where you were asked to implement a function that computes the matrix  $T_n^0$  using the Denavit-Hartenberg convention for an  $n$ -joint robot. You should be able to invoke your function using a syntax such as

```
T = dh(a, alpha, d, theta)
```

where `a`, `alpha`, `d`, and `theta` are the Denavit-Hartenberg parameters for the robot; the function name and order of parameters is not important.

Derive the Denavit-Hartenberg parameters using the frames shown below for the A150/A255 robots. Check the correctness of your Matlab implementation of the Denavit-Hartenberg convention by comparing the computed position of frame  $\{5\}$  (relative to frame  $\{0\}$ ) to the position reported by the robot's function `w0`. Write a Matlab script that moves the robot to 5 different locations spaced widely apart; at each location, output the position of frame  $\{5\}$  (relative to frame  $\{0\}$ ) computed by your function and that returned by `w0`.

Submit whatever Matlab files are needed to run your script using the `submit` command:

```
submit 4421 a1 <your files>
```

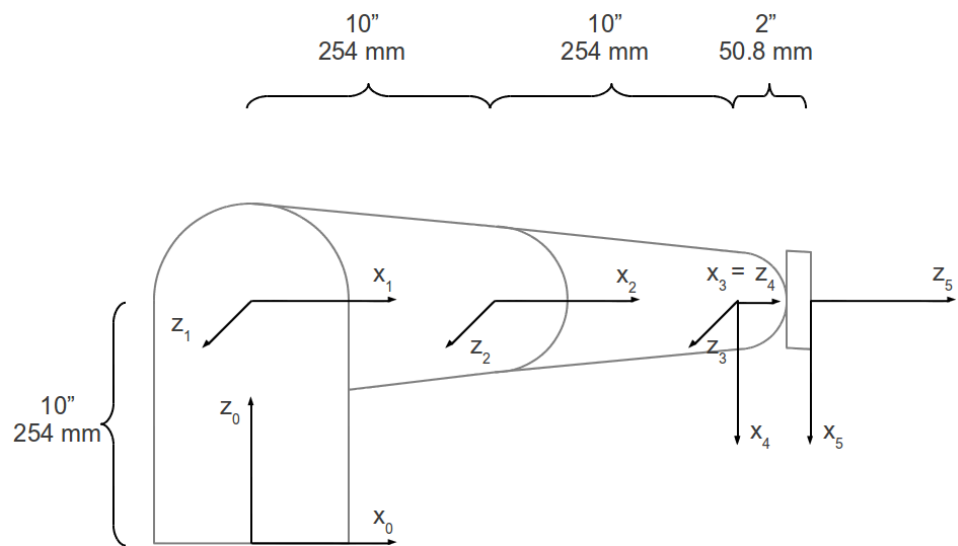


Figure 2: Frame locations for the A150 and A255 robots. The A150 uses dimensions in inches, and the A255 uses dimensions in millimeters.