York University CSE2001 June 4, 2012

# Homework Assignment #4
# Due: June 11, 2012 at 7:00 p.m.

**1.** Let $\Sigma = \{0, 1\}$. Let $L = \{uvw\#v : u, v, w \in \Sigma^*\}$. Is $L$ regular? Prove your answer is correct.

**2.** Let $\Sigma$ be an alphabet. We define a function $f$ that maps regular expressions to regular expressions as follows:

$$\begin{aligned}
f(\emptyset) &= \emptyset, \\
f(\varepsilon) &= \varepsilon, \\
f(a) &= a \text{ for } a \in \Sigma, \\
f(E_1E_2) &= f(E_2)f(E_1), \text{ where } E_1 \text{ and } E_2 \text{ are regular expressions,} \\
f(E_1 \cup E_2) &= f(E_1) \cup f(E_2), \text{ where } E_1 \text{ and } E_2 \text{ are regular expressions, and} \\
f(E^*) &= f(E)^*, \text{ where } E \text{ is a regular expression.}
\end{aligned}$$

**(a)** Write down the regular expression $f(a(ab)^* \cup \varepsilon)$.

**(b)** Prove that $L(f(E)) = L(E)^R$ for all regular expressions $E$.

In your solutions, you may use the following lemmas.

**Lemma 1** For any strings $y$ and $z$, $(yz)^R = z^R y^R$.
**Proof**: Let $\ell_y = |y|, \ell_z = |z|$ and $\ell = \ell_y + \ell_z$.

$$\begin{aligned}
(yz)^R[i] &= (yz)[\ell + 1 - i] && \text{(Defn reverse)} \\
&= \left\{ \begin{array}{ll} y[\ell + 1 - i] & \text{if } \ell + 1 - i \leq \ell_y \\ z[\ell + 1 - i - \ell_y] & \text{if } \ell + 1 - i > \ell_y \end{array} \right\} && \text{(Defn concatenation)} \\
&= \left\{ \begin{array}{ll} y[\ell_y + 1 - i + \ell_z] & \text{if } \ell + 1 - i \leq \ell_y \\ z[\ell_z + 1 - i] & \text{if } \ell + 1 - i > \ell_y \end{array} \right\} && \text{(since } \ell = \ell_y + \ell_z) \\
&= \left\{ \begin{array}{ll} y[\ell_y + 1 - i + \ell_z] & \text{if } i > \ell_z \\ z[\ell_z + 1 - i] & \text{if } i \leq \ell_z \end{array} \right\} && \text{(since } \ell = \ell_y + \ell_z) \\
&= \left\{ \begin{array}{ll} y^R[i - \ell_z] & \text{if } i > \ell_z \\ z^R[i] & \text{if } i \leq \ell_z \end{array} \right\} && \text{(Defn reverse)} \\
&= (z^R y^R)[i] && \text{(Defn concatenation)}
\end{aligned}$$

**Lemma 2** For any languages $L_1$ and $L_2$, $(L_1L_2)^R = L_2^R L_1^R$.
**Proof**: For any string $x$, we have:

$$\begin{aligned}
x \in (L_1L_2)^R \quad &\text{iff } \exists w \in L_1L_2 \text{ such that } x = w^R && \text{(Defn reverse)} \\
&\text{iff } \exists y \in L_1 \text{ and } z \in L_2 \text{ such that } x = (yz)^R && \text{(Defn concatenation)} \\
&\text{iff } \exists y \in L_1 \text{ and } z \in L_2 \text{ such that } x = z^R y^R && \text{(Lemma 1)} \\
&\text{iff } \exists u \in L_1^R \text{ and } v \in L_2^R \text{ such that } x = vu && \text{(take } u = y^R, v = z^R) \\
&\text{iff } x \in L_2^R L_1^R && \text{(Defn concatenation)}
\end{aligned}$$