MATH/CSE 1019 Final examination
Winter 2007
April 20, 2007
Instructor: S. Datta

LAST name: ______________________ FIRST name: ______________________

Student number: ______________________

Instructions:

1. If you have not done so, put away all books, papers, cell phones and pagers. Write your name and student number NOW!
2. Check that this examination has 14 pages. There should be 6 questions, each worth 20 points.
3. You have 180 minutes to complete the exam. Use your time judiciously.
4. Show all your work. Partial credit is possible for an answer, but only if you show the intermediate steps in obtaining the answer.
5. If you need to make an assumption to answer a question, please state the assumption clearly.
6. Points will be deducted for **vague and ambiguous** answers.
7. Your answers MUST be LEGIBLE.
8. Feel free not to use the hints supplied.

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

DO NOT WRITE ANYTHING ON THIS PAGE.

Q	part a	part b	part c	part d	part e	TOTAL
1						
2						
3						
4						
5						
6						

Aggregate score =

1. Logic and proofs.

   (a) (5 points) Given a proposition $P(x)$, express the following statement using propositional logic: "there exists a unique x in the domain such that $P(x)$ is true".

   (b) (4 points) Show that if a, b are real numbers and $a \neq 0$ then there is a unique real number r such that $ar + b = 0$.

   (c) (4 points) Give an example of a predicate $P(x, y)$ such that the following two statements are not logically equivalent:
   $\forall x \exists y P(x, y)$ and $\exists y \forall x P(x, y)$.

(d) (4 points) Prove that if n is an integer and $5n + 4$ is odd, then n is odd.

(e) (3 points) Prove that if a, b, c are positive real numbers, then the arithmetic mean $(a + b)/2$ is no smaller than the geometric mean $\sqrt{ab}$.

2. Sets.

   (a) (1+2+2 points) What is the power set $P(S)$ of a set S? Write down explicitly the sets $P(\phi)$ and $P(P(\phi))$, where ϕ is the empty set.

   (b) (3 points) Write down the cartesian product of the sets $\{1, 2\}$ and $\{3, 4\}$.

   (c) (2+1 points) Simplify the expression $\overline{A \cup B}$, using De Morgan's laws. Draw a Venn diagram to corroborate your answer.

(d) (2+4 points) Consider the set of positive integers Z^+. What is the set $Z^+ \times Z^+ \times Z^+ \times Z^+$? Is this set countable? Prove your answer.

(e) (3 points) Compute the following sum:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} ij.$$

3. Algorithms.

   (a) (2 points) Why are statements like $n^2 + n = O(n^2)$ mathematically meaningless?

   (b) (5 points) Consider the binomial coefficient $C(n, 3)$. Find the simplest possible function $f(n)$ such that $C(n, 3) \in \Theta(f(n))$, and prove your answer.

   (c) (3 points) If $f(n) \in O(g(n))$, is it true that $2^{f(n)} \in O(2^{g(n)})$? Prove your answer.

(d) (2+4 points) Consider the following pseudocode segment.

```
WEIRD(n)
if n = 1
  then return 1
  else r ← 1
       for i ← 1 to n − 1
       do r ← r * 2
       return (r + WEIRD(n − 1))
```

Justify that the value returned by WEIRD(n) is given by the recurrence
WEIRD(n) = WEIRD ($n-1$) $+2^{n-1}$.
Then prove that the value returned by WEIRD (n) is $2^n - 1$.

(e) (4 points) Consider the functions $2^{n \log_2 n}$ and n^n. Are they equal? If not, which is bigger? Prove your answer.

4. Induction and Recursion

   (a) (5 points) Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

   (b) (6 points) Use induction to prove that $a - b$ is a factor of $a^n - b^n$ whenever n is a positive integer. Hint: it may help to use the expression $(a - b)(a^n - b^n)$ in the inductive step.

(c) (5 points) Give a recursive definition for the set of all palindromes of even length over the alphabet $\{0, 1\}$. Recall that palindromes are strings that read the same forwards and backwards.

(d) (2 points) Give a recursive definition for the set of all odd positive integers.

(e) (2 points) Provide a simple (non-recursive) formulation for the following recursively defined function $f(1) = 1$ and for $n > 1$, $f(n) = 3f(n-1)$.

5. Counting - I

   (a) (3 points) Suppose you toss a coin 10 times. How many ways can you get an equal number of heads and tails?

   (b) (3 points) How many license plates consisting of 3 letters followed by 3 digits contain no letter or digit twice?

   (c) (5 points) Suppose n is an even positive integer. How many bit strings of length n are palindromes? Justify your answer.

(d) (4 points) A department contains 10 men and 15 women. How many ways can a 6-member committee be chosen if there are an equal number of women and men on the committee?

(e) (5 points) How many bit strings of length 10 over the alphabet $\{a, b, c\}$ have exactly 3 a's or exactly 4 b's?
Hint: this is not an exclusive-or.

6. Counting - II

   (a) (3 + 3 points) Consider the equation $x_1 + x_2 + x_3 = 17$. How many solutions does this have if $x_1 \geq 0$ and $x_2 \geq 0$ and $x_3 \geq 0$? What is the number of solutions if $x_1 \geq 1$ and $x_2 \geq 1$ and $x_3 \geq 1$?

   (b) (1+3 points) Suppose there are 90 students in this class. Consider the month in which the most students in this class were born. What is the minimum number of students that were born in this month? Prove your answer from first principles, i.e. without using a theorem/principle from the book.

(c) (3 points) Give a formula for the coefficient of x^8 in the expansion of $(x+1)^{10}$?

(d) (4 points) Give a formula for the coefficient of x^{10} in the expansion of $(x+1/x)^{20}$?
Hint: consider using the fact that $x + \frac{1}{x} = \frac{x^2+1}{x}$.

(e) (3 points) Prove the following equation.

$$\sum_{i=0}^{n} 3^i C(n,i) = 4^n.$$