CSE4421/5324: Assignment 1

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1. Find the homogeneous transformation T_1^0 where:

   (a) $\{1\}$ has the same orientation as $\{0\}$ and the origin of $\{1\}$ is translated relative to the origin of $\{0\}$ by $d_1^0 = \begin{bmatrix}1 & -1 & 2\end{bmatrix}^T$.

   (b) The origin of $\{1\}$ is coincident with the origin of $\{0\}$, and $\hat{x}_1^0 = \hat{y}_0^0$, $\hat{y}_1^0 = -\hat{z}_0^0$, and $\hat{z}_1^0 = -\hat{x}_0^0$.

   (c) The origin of $\{0\}$ is translated relative to the origin of $\{1\}$ by $d_0^1 = \begin{bmatrix}0 & 0 & -1.7321\end{bmatrix}^T$, and $\hat{x}_1^0 = \begin{bmatrix}0.7887 & -0.2113 & -0.5774\end{bmatrix}^T$, $\hat{y}_1^0 = \begin{bmatrix}-0.2113 & 0.7887 & -0.5774\end{bmatrix}^T$, and $\hat{z}_1^0 = \begin{bmatrix}0.5774 & 0.5774 & 0.5774\end{bmatrix}^T$.

2. Find the missing elements of the following rotation matrices. Show your work, or explain your reasoning. It may be the case that there is no unique solution, in which case you should find all possible solutions. *Hint: Consider using the cross product.*

   (a) $\begin{bmatrix} \cdot & 0 & -1 \\ \cdot & 0 & 0 \\ \cdot & -1 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} \sqrt{2}/2 & \cdot & 0 \\ \cdot & 0 & 1 \\ \cdot & -\sqrt{2}/2 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} \sqrt{2}/2 & 0 & \cdot \\ \sqrt{2}/2 & \cdot & \cdot \\ \cdot & \cdot & 0 \end{bmatrix}$

3. Consider the following 4×4 homogeneous transformation matrices:

$$
\begin{aligned}
R_{x,a} &: \text{rotation about } x \text{ by an angle } a \\
R_{y,a} &: \text{rotation about } y \text{ by an angle } a \\
R_{z,a} &: \text{rotation about } z \text{ by an angle } a \\
D_{x,a} &: \text{translation along } x \text{ by a distance } a \\
D_{y,a} &: \text{translation along } y \text{ by a distance } a \\
D_{z,a} &: \text{translation along } z \text{ by a distance } a
\end{aligned}
$$

   Write the matrix product giving the overall transformation for the following sequences (do not perform the actual matrix multiplications):

(a) The following rotations all occur in the moving frame.

i. Rotate about the current z-axis by angle ϕ.

ii. Rotate about the current y-axis by angle θ.

iii. Rotate about the current z-axis by angle ψ.

Note: This yields the ZYZ-Euler angle rotation matrix.

(b) The following rotations all occur in a fixed (world) frame.

i. Rotate about the world x-axis by angle ψ.

ii. Rotate about the world y-axis by angle θ.

iii. Rotate about the world z-axis by angle ϕ.

Note: This yields the roll, pitch, yaw (RPY) rotation matrix.

(c) The following transformations all occur in the moving frame.

i. Rotate about the current z-axis by angle θ_i.

ii. Translate along the current z-axis by a distance d_i.

iii. Translate along the current x-axis by a distance a_i.

iv. Rotate about the current x-axis by angle α_i.

Note: This is the Denavit-Hartenberg transformation matrix.

4. A rotation matrix in 3D is defined in terms of dot products. Prove the following statement: It does not matter what frame is used to compute the dot products as long as all of the dot products are computed using the same frame.

   In other words, prove that the dot product of two vectors $u \cdot v$ does not depend on the choice of coordinate frame. *Hint:* $u \cdot v = u^T v$

5. Prove that the length of a vector is unchanged by rotation; that is, prove that $\|v\| = \|Rv\|$ for every (3D) rotation matrix R.

6. **Graduate students only**

   A quaternion is another representation of rotation in 3D. The quaternion $Q = (q_w, q_x, q_y, q_z)$ can be thought of as being a scalar q_w and a vector $\vec{q} = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix}^T$. Given two quaternions $A = (a_w, \vec{a})$ and $B = (b_w, \vec{b})$, the quaternion product $C = AB$ is defined as

$$\begin{aligned} c_w &= a_w b_w - \vec{a} \cdot \vec{b} \\ \vec{c} &= a_w \vec{b} + b_w \vec{a} + \vec{a} \times \vec{b} \end{aligned}$$

   where $\vec{a} \times \vec{b}$ is the cross product of $\vec{a}$ and $\vec{b}$.

   (a) Show that $Q_I Q = Q Q_I = Q$ for every unit quaternion Q where $Q_I = (1, 0, 0, 0)$, i.e., Q_I is the identity quaternion.

   (b) The conjugate Q^* of a quaternion $Q = (q_w, \vec{q})$ is given by $Q^* = (q_w, -\vec{q})$. Show that $Q^* Q = Q Q^* = (1, 0, 0, 0)$, i.e., Q^* is the inverse of Q.

   (c) The quaternion $Q = (q_w, \vec{q})$ where $q_w = \cos\frac{\theta}{2}$ and $\vec{q} = \begin{bmatrix} k_x \sin\frac{\theta}{2} & k_y \sin\frac{\theta}{2} & k_z \sin\frac{\theta}{2} \end{bmatrix}^T$ represents the rotation of angle θ about the unit vector $\hat{k} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}^T$. A vector $\vec{p} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^T$ can be rotated using the quaternion product QPQ^* where P is the quaternion $(0, \vec{p})$. Show that this is true for a rotation of angle θ about the z-axis.

7. **Graduate students only**

   Consider a vector v that is rotated about a unit vector $\hat{k}$ (passing through the origin) by an angle θ to form a new vector v':

   $$v' = R_{k,\theta} v$$

   Derive Rodrigues' rotation formula,

   $$v' = v \cos\theta + (\hat{k} \times v) \sin\theta + \hat{k}(\hat{k} \cdot v)(1 - \cos\theta)$$

   Do not replicate the Wikipedia derivation; instead, use the rotation matrix for rotation about an axis $\hat{k}$ by an angle θ.

8. **Programming Question for everyone**

   If you have never used Matlab, then work through Chapters 1 and 2 of the *Getting Starting Guide*:

   `http://www.mathworks.com/help/pdf_doc/matlab/getstart.pdf`

   and read the document *Working with Functions in Files*

   `http://www.mathworks.com/help/techdoc/matlab_prog/f7-41453.html`

   before attempting this question.

   Create 6 Matlab functions corresponding to the 6 basic transformations from Question 3. Your functions should be named `rx`, `ry`, `rz`, `tx`, `ty`, and `tz` and they should return the appropriate homogeneous transformation matrix given an angle of rotation (in degrees) or a displacement; see the following Matlab script for examples.

```
% rotation about x by 10 degrees
R1 = rz(10);

% rotation about y by 20 degrees
R2 = ry(20);

% rotation about z by -30 degrees
R3 = rz(-30);

% translation along x by 1 unit
D4 = dx(1);

% translation along y by -1 unit
D5 = dy(-1);

% translation along z by 5 units
D6 = dz(5);
```

   You can find my implementation of `rz` here:

   `http://www.cse.yorku.ca/course/4421/assignments/rz.m`

   Using your newly created functions, write a short script named `commute.m` that illustrates the fact that a rigid transformation involving a rotation about and a translation along the same axis does not depend on the order of the rotation and translation (i.e., you can do the rotation followed by the translation or vice versa). Show that this is true using one example for each of the x, y, and z-axes.

   Submit your 7 Matlab files in a zip archive emailed to me.