Theorem 1.39 in the textbook states that if a language is accepted by an NFA, then it is also accepted by a DFA. (This uses the subset construction, where the DFA keeps track of all possible states that the NFA could be in.) The proof of Theorem 1.39 in the textbook gives the construction of the DFA, but does not prove that it is correct.

Here are the missing details (which were provided in the September 30 lecture). We assume the NFA has no ε-transitions. (If it does, first get rid of them, as described in the lecture.)

Let L be any language that is accepted by some NFA $M = (Q, \Sigma, \delta, q_0, F)$. We construct a DFA $M' = (Q', \Sigma, \delta', q_0', F')$, where

$$\begin{aligned} Q' &= \mathcal{P}(Q) = \text{the set of all subsets of } Q \\ q_0' &= \{q_0\} \\ F' &= \{S \subseteq Q : S \cap F \neq \emptyset\} = \text{the subsets of } Q \text{ that contain at least one element of } F \\ \delta'(S, a) &= \bigcup_{r \in S} \delta(r, a) = \{q \in Q : \exists r \in S \text{ such that } q \in \delta(r, a)\} \end{aligned}$$

First we want to show that the construction really implements the idea: that the state of M' really is the set of possible states of M after processing the same string. I'll introduce just one piece of notation for this: $\delta^*(x)$ is the state of M' after reading string $x \in \Sigma^*$. Formally, I can define this inductively by:

$$\begin{aligned} \delta^*(\varepsilon) &= q_0', \\ \delta^*(wa) &= \delta'(\delta^*(w), a) \text{ for } w \in \Sigma^*, a \in \Sigma. \end{aligned}$$

(Intuition behind the second equation: when processing wa, M' first processes w, which takes it to state $\delta^*(w)$ and then applies the δ' transition function once more when it processes the last character a.)

If we combine the definition of $\delta^*(wa)$ and the definition of δ', we get:

$$\delta^*(wa) = \delta'(\delta^*(w), a) = \{q \in Q : \exists r \in \delta^*(w) \text{ such that } q \in \delta(r, a)\} \tag{1}$$

Now we're ready to prove the key claim (the invariant of the construction).

Claim: There is a path from q_0 to q in M labelled by string x if and only if $q \in \delta^*(x)$.

Proof by induction on the length of x:

Base case $|x| = 0$. Then $x = \varepsilon$.

There is a path from q_0 to q in M labelled by ε
$\Leftrightarrow q = q_0$ (since a path of length 0 starts and ends at the same place)
$\Leftrightarrow q \in \{q_0\} = q_0' = \delta^*(\varepsilon)$ (by the definition of q_0' and $\delta^*(\varepsilon)$).

Inductive step. Let $k \geq 0$. Assume the claim is true for all strings x of length k. We shall prove the claim is true for all strings x of length $k + 1$.

Let x be any string of length $k + 1$. Then $x = wa$ where w is a string of length k and a is a single character. Then,

There is a path from q_0 to q in M labelled by x
$\Leftrightarrow$ There is a path from q_0 to q in M labelled by wa (since $x = wa$)
$\Leftrightarrow$ There is a state $r \in Q$ such that there is a path from q_0 to r in M labelled by w and $q \in \delta(r, a)$
$\Leftrightarrow$ There is a state $r \in \delta^*(w)$ and $q \in \delta(r, a)$ (by the induction hypothesis)
$\Leftrightarrow q \in \delta^*(wa)$ (by Equation (1)).

This completes the inductive proof of the claim. Now we show that the two machines really accept exactly the same strings.

Claim: M accepts a string x if and only if M' accepts x.

Proof: For any string x,

M accepts x
$\Leftrightarrow$ There is a path in M labelled by x that goes from state q_0 to some state of F

$\Leftrightarrow$ Some state of F is in $\delta^*(x)$ (by the previous claim)
$\Leftrightarrow$ The state of M' after reading x contains an element of F (by the definition of δ^*)
$\Leftrightarrow$ The state of M' after reading x is an accepting state of M' (by the definition of F')
$\Leftrightarrow$ M' accepts x.

This completes the proof that any language accepted by an NFA is also accepted by a DFA. The converse is trivial since a DFA is a special case of an NFA; thus, if a language is accepted by a DFA then it is also accepted by an NFA.

This proves that the class of languages accepted by NFAs is the same as the class of regular languages.