Solutions to Homework Assignment #5

1.

(a) For $n \geq 2$ we have:

$$\begin{aligned} b_n &= na_n \\ &= n((1 - \frac{1}{n})a_{n-1} + 2n) \\ &= (n-1)a_{n-1} + 2n^2 \\ &= b_{n-1} + 2n^2. \end{aligned}$$

(b) **The Easy Way**: From the format of the recurrence, it's pretty easy to guess the following claim.

Claim: $b_n = \sum_{i=2}^{n} 2i^2$.

Base case ($n = 1$): $b_1 = 0$ and the sum is the sum of 0 terms.

Induction step: Let $n > 1$. Assume $b_{n-1} = \sum_{i=2}^{n-1} 2i^2$. Then,

$$\begin{aligned} b_n &= b_{n-1} + 2n^2 \\ &= \left(\sum_{i=2}^{n-1} 2i^2\right) + 2n^2 \\ &= \sum_{i=2}^{n} 2i^2. \end{aligned}$$

In class we found a formula for the sum of consecutive squares. Using that, we get

$$\begin{aligned} b_n &= 2\sum_{i=2}^{n} i^2 \\ &= 2\left(\sum_{i=1}^{n} i^2\right) - 2 \\ &= 2 \cdot \frac{n(n+1)(2n+1)}{6} - 2 \\ &= \frac{n(n+1)(2n+1)}{3} - 2. \end{aligned}$$

The Hard Way: We can also solve this problem using Theorem 6 on page 421. The characteristic polynomial of the recurrence is $r - 1$, which has 1 as its single root. We have $F(n) = 2n^2 = (2n^2 + 0n + 0) \cdot 1^n$. Since 1 is a root of the characteristic polynomial, the theorem says there is a solution to the non-homogeneous recurrence of the form

$b_n = n^1(p_2n^2 + p_1n + p_0) = p_2n^3 + p_1n^2 + p_0n$ (where p_0, p_1, p_2 are some constants). To find the constants that work, we plug this into the recurrence:

$$\begin{aligned} p_2n^3 + p_1n^2 + p_0n &= p_2(n-1)^3 + p_1(n-1)^2 + p_0(n-1) + 2n^2 \\ &= p_2n^3 - 3p_2n^2 + 3p_2n - p_2 + p_1n^2 - 2p_1n + p_1 + p_0n - p_0 + 2n^2, \text{ so} \\ 0 &= (-3p_2 + 2)n^2 + (3p_2 - 2p_1)n + (-p_2 + p_1 - p_0). \end{aligned}$$

Since this must be true for all n, we must have

$$\begin{aligned} 3p_2 &= 2 \\ -2p_1 + 3p_2 &= 0 \\ -p_0 + p_1 - p_2 &= 0 \end{aligned}$$

Solving these 3 linear equations in 3 unknowns, we get $p_0 = \frac{1}{3}, p_1 = 1, p_2 = \frac{2}{3}$. So a particular solution to the non-homogeneous recurrence is $b_n = \frac{2}{3}n^3 + n^2 + \frac{1}{3}n$.

The homogeneous version of the recurrence is $b_n = b_{n-1}$, and every solution to this recurrence is of the form $b_n = \alpha$ (where α is a constant). Thus, by Theorem 5 on page 420, solutions to the non-homogeneous version of the recurrence are of the form $b_n = \frac{2}{3}n^3 + n^2 + \frac{1}{3}n + \alpha$. To figure out the constant α, we use the initial condition of the recurrence: $b_1 = 0$. This means $\frac{2}{3} + 1 + \frac{1}{3} + \alpha = 0$, so $\alpha = -2$. Thus, the solution to the recurrence is $b_n = \frac{2}{3}n^3 + n^2 + \frac{1}{3}n - 2$.

(c) Since $a_n = b_n/n$, the solution for the recurrence is $a_n = \frac{2}{3}n^2 + n + \frac{1}{3} - \frac{2}{n}$.

2. First, let us find a particular solution to the equation

$$a_n = 2a_{n-1} + 4a_{n-2} - 8a_{n-3} + 9n. \tag{1}$$

The characteristic polynomial is $r^3 - 2r^2 - 4r + 8 = (r-2)^2(r+2)$, which has roots 2 and -2. By Theorem 6 on page 421, there should be a particular solution of the recurrence equation (1) that is a linear function. So let's try plugging $a_n = cn + d$, where c and d are some constants, into the recurrence to find values of c and d that work.

$$\begin{aligned} cn + d &= 2(c(n-1) + d) + 4(c(n-2) + d) - 8(c(n-3) + d) + 9n \\ cn + d &= 2cn - 2c + 2d + 4cn - 8c + 4d - 8cn + 24c - 8d + 9n \\ cn + d &= -2cn + 14c - 2d + 9n \\ 0 &= (9 - 3c)n + (14c - 3d) \end{aligned}$$

These equations can be satisfied for all n by choosing $c = 3$ and $d = 14$. Thus, $a_n = 3n + 14$ is a particular solution to equation (1).

Now, we need to find a solution that also satisfies the initial conditions. Recall that the characteristic polynomial is $(r-2)^2(r+2)$, so by Theorem 4 on page 418, every sequence of the form $a_n = \alpha 2^n + \beta n 2^n + \gamma(-2)^n$ (where α, β and γ are constants) satisfies the homogeneous version of the recurrence (*i.e.* $a_n = 2a_{n-1} + 4a_{n-2} - 8a_{n-3}$). By Theorem 5 on page 420,

$a_n = \alpha 2^n + \beta n 2^n + \gamma(-2)^n + 3n + 14$ is a solution to equation (1). So if we find values of α, β and γ that satisfy the initial conditions, we are done.

We must have

$$\begin{aligned} \alpha + \gamma + 14 &= a_0 = 14, \\ 2\alpha + 2\beta - 2\gamma + 17 &= a_1 = 31, \text{ and} \\ 4\alpha + 8\beta + 4\gamma + 20 &= a_2 = 28. \end{aligned}$$

Solving these three linear equations, we get $\alpha = 3, \beta = 1$ and $\gamma = -3$. Thus, the required solution is $a_n = 3 \cdot 2^n + n2^n - 3(-2)^n + 3n + 14$.

3. For this question, we simply apply the Master Theorem on page 430.

(a) $f(n)$ is $O(n)$. (Here, $a = 2, b = 3, c = 5, d = 1$ and $b^d = 3 > 1 = a$.)

(b) $f(n)$ is $O(n^2 \log n)$. (Here, $a = 4, b = 2, c = 8, d = 2$ and $b^d = 4 = a$.)

4.

(a) If $((1,2),(c,d))$ is in R, then $1d = 2c$, so any pairs (c,d) that have $d = 2c$ (and $d \neq 0$) would be correct answers for this part. For example, $(1,2), (2,4), (3,6), (-1,-2)$.

(b) Take any $(a,b) \in Q$. Then $ab = ba$, so $((a,b),(a,b)) \in R$.

(c) Suppose $((a,b),(c,d)) \in R$. Then $ad = bc$, so $cb = da$. Thus, $((c,d),(a,b))$ is also in R.

(d) Suppose $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$. Then, $ad = bc$ and $cf = de$. It follows that $adf = bcf = bde$. Since $d \neq 0$, we have $af = be$, so $((a,b),(e,f)) \in R$.

(Remark: Formally speaking, the set of rational numbers is usually defined as the set of equivalence classes using this equivalence relation on Q.)

5. Note: All of the theorems about linear homogeneous equations we saw in class work equally well if the roots of the characteristic polynomial are complex numbers. (Nothing in the proofs used any property of real numbers that is not also true of complex numbers.)

The characteristic polynomial of this recurrence is $r^4 - r^2 + 2r + 2 = (r+1)(r^3 - r^2 + 2) = (r+1)^2(r^2 - 2r + 2) = (r+1)^2(r - \frac{2+\sqrt{4-8}}{2})(r - \frac{2-\sqrt{4-8}}{2}) = (r+1)^2(r-(1+i))(r-(1-i))$, where $i = \sqrt{-1}$. So, Theorem 4 on page 418 says that $a_n = (\alpha n + \beta)(-1)^n + \gamma(1+i)^n + \delta(1-i)^n$ is a solution to the recurrence relation, for any constants $\alpha, \beta, \gamma, \delta$. We now choose the constants to satisfy they initial conditions. We must have:

$$\begin{aligned} \beta + \gamma + \delta &= 0 \\ -\alpha - \beta + (1+i)\gamma + (1-i)\delta &= 0 \\ 2\alpha + \beta + 2i\gamma - 2i\delta &= -12 \\ -3\alpha - \beta + (-2+2i)\gamma + (-2-2i)\delta &= 28 \end{aligned}$$

Solving these linear equations gives $\alpha = -8, \beta = 4, \gamma = -2, \delta = -2$, so the solution to the recurrence is $a_n = (-8n+4)(-1)^n - 2(1+i)^n - 2(1-i)^n$.

We can further simplify this using some high-school algebra and trigonometry:

$$\begin{aligned}
a_n &= (-8n+4)(-1)^n - 2((1+i)^n + (1-i)^n) \\
&= (-8n+4)(-1)^n - 2((\sqrt{2}i^{\pi/4})^n + (\sqrt{2}i^{-\pi/4})^n) \\
&= (-8n+4)(-1)^n - 2(2^{n/2}i^{n\pi/4} + 2^{n/2}i^{-n\pi/4}) \\
&= (-8n+4)(-1)^n - 2^{n/2+1}(i^{n\pi/4} + i^{-n\pi/4}) \\
&= (-8n+4)(-1)^n - 2^{n/2+1}(\cos(n\pi/4) + i\sin(n\pi/4) + \cos(-n\pi/4) + i\sin(-n\pi/4)) \\
&= (-8n+4)(-1)^n - 2^{n/2+2}\cos(n\pi/4).
\end{aligned}$$