Homework Assignment #4
Due: March 17, 2:30 p.m.

The front page of your solution set should be a cover page that includes only the following: your name, your student number, a list of students with whom you have discussed the problems, and a signed declaration stating "I have read and understood the policy on academic honesty on the course web page". Without this declaration, your solutions will not be marked.

1. Let n be a positive integer. Suppose there are n people and a pile containing n different pairs of gloves.

(a) How many ways can the n people put on all the gloves? (It doesn't matter who chooses gloves first or second; all that matters is the outcome when everybody has put on the gloves, *i.e.* who is wearing which gloves.) Briefly explain why your answer is correct.

(b) How many of the ways in part (a) have at least one person with a mismatched pair of gloves? Briefly explain why your answer is correct.

(Remark: you can gain some confidence in your answers by testing if they are correct for some small values of n: just list all possible ways. Don't hand these lists in, though. Of course, the explanation you do hand in should be general and apply for all values of n.)

2. Let n be a positive integer. Suppose there are n men and n women. All of them are unmarried. None of them are related to one another. They wish to have a big wedding ceremony where they all get married.

(a) How many ways can they be matched up into n married couples in Canada? (What matters here is who marries whom. Ordering of the couples is unimportant. Similarly order within the couples is unimportant: John marrying Jane is the same as Jane marrying John.) Briefly explain why your answer is correct.

(b) How many ways can they be matched up into n married couples in Florida, where same-sex marriage is still not legal? Briefly explain why your answer is correct.

(c) Answer part (a) again, but this time assume that two of the men are brothers of one of the women. There are no other family relationships between the people.

(d) Answer part (b) again, but this time assume that two of the men are brothers of one of the women. There are no other family relationships between the people.

3. A computer programme generates a sequence of n passwords. Each password is a string of 4 characters, where each character is either an upper-case letter, a lower-case letter or a digit $(0, 1, \ldots, 9)$. Each password contains at least 1 upper-case letter and at least 1 digit.

What is the smallest value of n that guarantees at least 5 of the generated passwords will be identical? (Your answer should be a number.) Explain why your answer is correct.

4. Willemina wants to encode a string of digits $(0, 1, \ldots, 9)$ using a string of bits. She decides to use the following encoding scheme:

digit	code
0	00
1	01
2	1000
3	10010
4	10011
5	101
6	110
7	11100
8	11101
9	1111

So, for example, she encodes the string 3147 as 10010011001111100. Not every bit string is a legal encoding. For example, 01100 does not encode any string of digits. However, each bit string is the encoding of at most one string of digits (this is because none of the 10 bit strings used to encode a digit is a prefix of any of the others).

Let a_n be the number of different strings of digits whose encodings are exactly n bits long. Give a recurrence relation for a_n, including the initial conditions. Briefly explain why your recurrence is correct. You do not have to solve the recurrence relation.

5. Solve the following recurrence relation:

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 9 \\ a_2 &= 11 \\ a_n &= a_{n-1} + 8a_{n-2} - 12a_{n-3}, \text{ for } n \geq 3. \end{aligned}$$