CSE 3101: Solution for Assignment 1

total: 45

Question 1 [6] Let c be a positive constant, $c \neq 1$. Prove the following equalities by induction on n, for $n \geq 1$. (Note that when $n = 1$, the LHS of both equalities are just 1.)

a)

$$1 + c + c^2 + \ldots + c^{n-1} = \frac{c^n - 1}{c - 1}$$

b)

$$1 + 2c + 3c^2 + \ldots + nc^{n-1} = \frac{nc^{n+1} - (n+1)c^n + 1}{(c-1)^2}$$

Solution 1a [3pt]

Base case n = 1. Then both LHS and RHS are 1, so LHS = RHS.

Induction step Assume that the equality hold for n. We show that it is also true for $n + 1$. For $n + 1$, the LHS is

$$\begin{aligned} & 1 + c + c^2 + \ldots + c^{n-1} + c^n \\ = \quad & (1 + c + c^2 + \ldots + c^{n-1}) + c^n \\ = \quad & \frac{c^n - 1}{c - 1} + c^n \qquad \text{by I.H.} \\ = \quad & \frac{c^n - 1 + (c-1)c^n}{c - 1} \\ = \quad & \frac{c^{n+1} - 1}{c - 1} \end{aligned}$$

Q.E.D.

Solution 1b [3pt]

Base case n = 1. Then both LHS and RHS are 1, so LHS = RHS.

Induction step Assume that the equality hold for n. We show that it is also true for $n + 1$. For $n + 1$, the LHS is

$$\begin{aligned} & 1 + 2c + 3c^2 + \ldots + nc^{n-1} + (n+1)c^n \\ = \quad & (1 + 2c + 3c^2 + \ldots + nc^{n-1}) + (n+1)c^n \\ = \quad & \frac{nc^{n+1} - (n+1)c^n + 1}{(c-1)^2} + (n+1)c^n \qquad \text{by I.H.} \\ = \quad & \frac{nc^{n+1} - (n+1)c^n + 1 + (c-1)^2(n+1)c^n}{(c-1)^2} \\ = \quad & \frac{nc^{n+1} - (n+1)c^n + 1 + (n+1)(c^{n+2} - 2c^{n+1} + c^n)}{(c-1)^2} \\ = \quad & \frac{(n+1)c^{n+2} - (n+2)c^{n+1} + 1}{(c-1)^2} \end{aligned}$$

Q.E.D.

Question 2 [20] The following recurrence often arises in analyzing the time complexity of recursive algorithms:

$$S(n) = aS(n-1) + g(n) \qquad \text{for } n \geq 2 \tag{1}$$

where $a > 0$ and $S(1) > 0$ are some constants.

In parts **a)**, **b)** and **c)** below we consider the case where $g(n) = d^n$, for some constant $d > 0$. Then we will consider the case where $g(n) = cna^n$ in parts **d)** and **e)**.

First, suppose that $g(n) = d^n$, for some constant $d > 0$. We can "unwind" $S(n)$ as follows:

$$\begin{aligned}
S(n) &= aS(n-1) + d^n \\
&= a(aS(n-2) + d^{n-1}) + d^n \\
&= a^2S(n-2) + [ad^{n-1} + d^n] \\
&= a^2(aS(n-3) + d^{n-2}) + [ad^{n-1} + d^n] \\
&= a^3S(n-3) + [a^2d^{n-2} + ad^{n-1} + d^n] \\
&\dots \\
&= a^{n-1}S(1) + [a^{n-2}d^2 + a^{n-3}d^3 + \dots + ad^{n-1} + d^n]
\end{aligned} \tag{2}$$

a) When $d = a$, show that $S(n) = \Theta(na^n)$.

Now consider the case where $d \neq a$. Write (2) as

$$S(n) = a^{n-1}S(1) + a^{n-2}d^2 \left[1 + \frac{d}{a} + \left(\frac{d}{a}\right)^2 + \dots + \left(\frac{d}{a}\right)^{n-2}\right] \tag{3}$$

Use the result from Question 1 **a)** to show that

b) If $a > d$ then $S(n) = \Theta(a^n)$.
c) If $d > a$ then $S(n) = \Theta(d^n)$.

Now consider the recurrence (1) where $g(n) = cna^n$, for some constant $c > 0$.

d) Expand $S(n)$ in the style of (2).
e) Show that $S(n) = \Theta(n^2a^n)$.

Solution 2a [3pt] When $d = a$, (2) becomes

$$\begin{aligned}
S(n) &= a^{n-1}S(1) + (n-1)a^n \\
&= (n - 1 + \frac{S(1)}{a})a^n \\
&= (n + K)a^n
\end{aligned}$$

where K is the constant $\frac{S(1)}{a} - 1$. (K can be negative.)

We want $c_1, c_2 > 0$ and $n_0 > 0$ so that

$$c_1na^n < (n + K)a^n < c_2na^n$$

for all $n > n_0$. That is

$$c_1n < n + K < c_2n$$

for all $n > n_0$.

Take $c_1 = 1/2$, $c_2 = 2$ and $n_0 = 2 + 2\lceil |K| \rceil$ ($|K|$ is the absolute value of K). Then for $n \geq n_0$,

$$\begin{aligned} n + K &= \frac{1}{2}n + \frac{1}{2}n + K \\ &\geq \frac{1}{2}n + 1 + \lceil |K| \rceil + K > \frac{1}{2}n \end{aligned}$$

(This shows the first inequality.) And

$$2n \geq n + 2 + 2\lceil |K| \rceil \geq n + |K|$$

(showing the second inequality). Q.E.D.

Solution 2b [5pt] Note that for $a > d$, $\frac{d}{a} < 1$, so $(\frac{d}{a})^n$ gets sufficiently small (close to 0) when n is sufficiently large.

Apply Question 1a for $c = \frac{d}{a}$, then (3) gives us

$$\begin{aligned} S(n) &= a^{n-1}S(1) + a^{n-2}d^2\frac{1 - c^{n-1}}{1 - c} \\ &= a^n\left[\frac{S(1)}{a} + \frac{d^2}{a^2}\frac{1 - c^{n-1}}{1 - c}\right] \\ &= a^nK(n) \end{aligned}$$

where

$$K(n) = \frac{S(1)}{a} + \frac{d^2}{a^2}\frac{1 - c^{n-1}}{1 - c}$$

As $n \longrightarrow \infty$, $K(n)$ approach a constant, so $S(n) = \Theta(a^n)$.

The proof (from definition of the Θ notation) is as follows: We have (since $c < 1$)

$$K(n) < \frac{S(1)}{a} + \frac{d^2}{a^2}\frac{1}{1 - c}$$

so we can take

$$c_2 = \frac{S(1)}{a} + \frac{d^2}{a^2}\frac{1}{1 - c}$$

(note that c_2 does not depend on n).

Also, for n large enough, $c^{n-1} < 1/2$, so

$$K(n) > \frac{S(1)}{a} + \frac{d^2}{a^2}\frac{1/2}{1 - c}$$

And we can take

$$c_1 = \frac{S(1)}{a} + \frac{d^2}{a^2}\frac{1/2}{1 - c}$$

(c_1 does not depend on n.) The "threshold" n_0 is such that $c^{n_0-1} < 1/2$ (so that $c^{n-1} < 1/2$ for all $n > n_0$). Q.E.D.

Solution 2c [5pt] Suppose that $d > a$, then $(\frac{d}{a}) > 1$. Apply Question 1a for $c = \frac{d}{a}$, we get from (3):

$$\begin{aligned}
S(n) &= a^{n-1}S(1) + a^{n-2}d^2 \ \frac{c^{n-1}-1}{c-1} \\
&= a^{n-1}S(1) + d^n(1/c)^{n-2} \ \frac{c^{n-1}-1}{c-1} \qquad \text{since } d^{n-2}(1/c)^{n-2} = a^{n-2} \\
&= a^{n-1}S(1) + d^n \ \frac{c-(1/c)^{n-2}}{c-1} \\
&= a^{n-1}S(1) + d^n \ \frac{1-(1/c)^{n-1}}{1-(1/c)} \\
&= d^n\frac{1}{a}(\frac{1}{c})^n S(1) + d^n \ \frac{1-(1/c)^{n-1}}{1-(1/c)} \\
&= d^n\left[\frac{1}{a}(\frac{1}{c})^n S(1) + \frac{1-(1/c)^{n-1}}{1-(1/c)}\right]
\end{aligned}$$

So $S(n) = d^n K'(n)$, where

$$K'(n) = \frac{1}{a}(\frac{1}{c})^n S(1) + \frac{1-(1/c)^{n-1}}{1-(1/c)}$$

Since $1/c < 1$, for $n \longrightarrow \infty$ we have $(1/c)^n \longrightarrow 0$, so $K'(n) \longrightarrow \frac{1}{1-(1/c)}$. Therefore for any $c_1 < \frac{1}{1-(1/c)}$ (such as $c_1 = \frac{1}{2}\frac{1}{1-(1/c)}$), and $c_2 > \frac{1}{1-(1/c)}$ (such as $c_2 = 2\frac{1}{1-(1/c)}$), there is a threshold n_0 so that for all $n > n_0$,

$$c_1 < K'(n) < c_2$$

Hence,

$$c_1 d^n < S(n) < c_2 d^n$$

Q.E.D.

Solution 2d [3pt] We expand $S(n)$ as follows:

$$\begin{aligned}
S(n) &= aS(n-1) + cna^n \\
&= a(aS(n-2) + c(n-1)a^{n-1}) + cna^n \\
&= a^2S(n-2) + [c(n-1)a^n + cna^n] \\
&= a^2(aS(n-3) + c(n-2)a^{n-2}) + [c(n-1)a^n + cna^n] \\
&= a^3S(n-3) + [c(n-2)a^n + c(n-1)a^n + cna^n] \\
&\dots \\
&= a^{n-1}S(1) + [c2a^n + c3a^n + \ldots + c(n-1)a^n + cna^n] \\
&= a^{n-1}S(1) + ca^n[2 + 3 + \ldots + n] \\
&= a^{n-1}S(1) + ca^n\left[\frac{n(n+1)}{2} - 1\right] \qquad \text{since } \sum_{i=1}^{n} = n(n+1)/2
\end{aligned}$$

Solution 2e [4pt] For $n > 2$ we have

$$\frac{n(n+1)}{2} - 1 > \frac{n^2}{2}$$

so

$$S(n) > \frac{c}{2} n^2 a^n$$

Also, take n_0 large enough so that $S(1) < acn^2$ for $n > n_0$. Then, since

$$\frac{n(n+1)}{2} - 1 < n^2$$

we have for $n > n_0$:

$$a^{n-1} S(1) < a^{n-1} acn^2 = cn^2 a^n$$

and

$$ca^n \left[\frac{n(n+1)}{2} - 1 \right] < cn^2 a^n$$

So

$$S(n) < 2cn^2 a^n$$

Now take $c_1 = \frac{c}{2}$ and $c_2 = 2c$, we have that for $n > max\{2, n_0\}$:

$$c_1 n^2 a^n < S(n) < c_2 n^2 a^n$$

So $S(n) = \Theta(n^2 a^n)$. Q.E.D.

Question 3 [19] In this question you are asked to come up with an array of 11 natural numbers which is somewhat specific to you, and then run the Heapsort algorithm on it. Consider the following array A with missing values for $A[1], A[2], A[7], A[8], A[11]$.

			20	6	16	5			31	12	
A	1	2	3	4	5	6	7	8	9	10	11

a [1pt] Write down the complete array A where

- $A[1]$ and $A[2]$ are the day and month of the birthday of one close friend of yours.
- $A[7]$ is the sum of the digits in your student number.
- $A[8]$ is the sum of the digits in your year of birth.
- $A[11]$ is the sum of the numbers for the day and month of your birthday.

b [2pt] Draw the binary tree with 11 nodes, whose nodes are labeled with $A[1], \ldots, A[11]$, so that the two childrens of the node $A[i]$ are nodes $A[2i]$ and $A[2i+1]$, for $1 \le i \le 5$.

c [10pt] Now we follows the Build-Heap procedure on A. Show the run of Build-Heap on A by drawing, for each $i \le 5$, the subtree at $A[i]$ as it is being modified during the call to *Heapify*(A, i).

d [2pt] Write down the updated array A. Exchange $A[1]$ and $A[11]$. Write down the resulted array.

e) Follow the Heapsort algorithm. Now the new heap-size is 10, and we need to consider only the sub-array $A[1], \ldots, A[10]$.

(i) [1pt] Draws the new heap (of size 10).
(ii) [2pt] Show how the heap changes during the run of *Heapify*$(A, 1)$.
(iii) [1pt] Write down the updated array A.