

Shape Representation and Recognition from Curvature

Gregory Dudek
Research Centre for Intelligent Machines
McGill University
Montreal, Que, Canada H3A 2A7
dudek@mrcrim.mcgill.edu

John K. Tsotsos
Department of Computer Science
University of Toronto
Toronto, Ont, Canada M5S 1A4
tsotsos@vis.toronto.edu

Abstract

This paper deals with a new part decomposition of curved objects. It provides a unified framework for both the decomposition and recognition of both planar curves as well as surfaces in three-dimensional space.

The decomposition operation simultaneously performs data interpolation, data smoothing, and segmentation. The unification of these three stages results in a smoothing operation that is coupled with the primitives to be used in description. Each of the minimization operators, in addition to having a curvature tuning, also has a different spatial sensitivity function. As a result, the different possible descriptions capture information at multiple spatial scales. This allows a single region of an object to be described in more than one way, when appropriate.

The practicality of the ensuing representation is demonstrated by the robust recognition of planar curves. A matching strategy based in dynamic programming is used. The results illustrate the manner in which a continuous spectrum of similar objects can be defined, ranging from those that are very similar to the target to those that are very different.

1 Introduction

In this paper, an approach is developed for describing objects for the purposes of recognition. We deal here with two key issues: building natural descriptions of curved objects, and making these descriptions compact and abstract. Typical example of the types of curve we are able to describe as both qualitatively similar, yet discriminably different, are shown in figure 1. Methods based on curvature extrema alone are likely to find three of the four of these shapes indistinguishable, while methods based on approaches such as shape templates may be oblivious to their similarity.

The stable extraction and measurement of curvature information in the presence of noise has been dealt with in several ways [6, 3, 11, 15]. One characteristic of most existing curvature-measurement techniques is the assumption that there is a unique curvature that can be measured at each point. While this is true in the analytic case, the assumption introduces significant problems for inverse problems involving noisy signals, such as those that occur in vision. Despite the respectable results that have been achieved by some researchers, the need for scale-specific operators to deal with noise problems causes an inherent preference for certain ranges of curvature value and involves strong implicit assumptions about the underlying signal. The actual curvature of a signal depends on what we call noise and what we call signal, and hence may take on differing values depending on our goals.

The central ideas of the approach described here are the following:

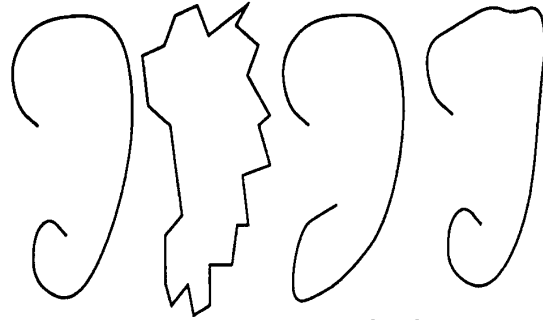


Figure 1: Discriminable, yet related, curves

- To use curvature as a fundamental descriptive attribute.
- To provide a multi-scale representation that can deal with regions that have more than one type of character.
- To target the smoothing technique to the particular model set.
- To handle noise in a natural and automatic manner.
- To obtain a concise, abstract description of objects.
- To obtain a representation that can be computed in parallel.

The curvature-tuned smoothing method applied in this paper allows us to obtain measurements which have not been subjected to an unnatural "flattening" or distortion as a result of the smoothing. It allows regions with different curvatures to be extracted as alternative descriptors at the same location. Since curvature and scale are intimately related, this provides a robust and multi-scale description of objects in the world. In this work we presuppose the availability of a set of points defining a plane curve $d(t)$, or points in 3-space defining a surface $d(x, y)$. Curvature ($\kappa(t)$ for a curve) appears to have both desirable computational as well as perceptual characteristics [2, 10, 1].

The partitioning of the set of curved subparts (i.e. the "curvature space") obtained from this method can be made far more finely than alternative ones based on solely the sign of curvature (or signs of mean and Gaussian curvature on surfaces on surfaces). This allows for more precise shape discrimination than is usually associated with curvature based object recognition, particularly in the case of three-dimensional surfaces.

Traditional approaches to the measurement of curvature for plane curves involve the use of non-local models.

The extent and shape of the neighborhood used for processing asserts an implicit scale specificity. The method described here exploits the relationship between curvature and scale to produce a set of *alternative* descriptions of the data based of structure at different scales.

2 Curvature, Ambiguity and Smoothing

In finding a smooth description of input data, the discontinuities that are inferred and the curvature estimates produced are a function of the smoothing model used. Furthermore, the smoothing model has a profound influence on the estimates of the underlying object's curvature. In performing smoothing and interpolation, even if we could construct a smoothing constraint which was not biased towards particular curvatures and scales we would be forced to select a single interpretation of the curve and its curvature at each point. The non-linear thin plate functional

$$E(u) = \int_{t_a}^{t_b} \|u(t) - d(t)\|^2 + \lambda \kappa(u(t), t)^2 dt \quad (1)$$

selects the lowest curvature interpretation of the original data while rejecting alternative (higher RMS curvature) interpretations even though they are equally plausible based on either physical or perceptual constraints (for example, see figure 2).

[bp]

Are minimum curvature approximations of the input data the most desirable natural interpretations? It is certainly true that curves and surfaces with small curvatures appear smooth and regular. These, however, are far from the only surfaces with this perceptual property. Likewise, regularizing stabilizers tuned for zero curvature are only a small subset of the large range of feasible stabilizing functionals that could be used. The customary preference for surfaces of near zero curvature, despite its mathematical simplicity, may be unnecessarily conservative as a model of either the natural world or human perception. In fact, when the surface data is intrinsically curved, the use of the conventional thin beam smoothing process (or any functional with a linear Green's function) may make it difficult to smooth noisy surface data sufficiently to extract measurements such as curvature



Figure 2: Ambiguity of "natural" descriptions. Is the data on top "really" a set of straight lines, as shown on the left by the thick lines, or a circular arc, as shown on the right?

without seriously corrupting them.

For example, the thin-plate smoothing kernel for a bi-infinite plane has a smoothing kernel is given by

$$G(t, t_0) = \frac{1}{2\lambda} e^{-\frac{|t-t_0|}{\lambda\sqrt{2}}} \cos\left(\frac{|t-t_0|}{\lambda\sqrt{2}} - \pi/4\right) \quad (2)$$

[5] which over large distances can be approximated by its envelope, a symmetric exponential decay, $G(\alpha, t) = e^{-\alpha|t_0-t|}$, with $\alpha = 1/\lambda\sqrt{2}$. This is a low-pass filter whose attenuation can be expressed as the function $R(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}$ in the frequency domain, that is, asymptotically proportional to spatial frequency. For sinusoidal input data $d(t) = A \sin(\frac{1}{A}t)$ with curvature $\kappa(t)$ the curvature, after smoothing with $ge^{-\alpha t}$ to reduce noise, is distorted to be

$$\kappa(G(\alpha) * d(t)) = \left\{ \frac{-g\alpha^2}{\alpha^2 + \frac{1}{A^2}} \right\} \kappa(d(t)) \quad (3)$$

This will apply similarly to any linear filter since a sinusoid is the eigenfunction for linear filtering. Hence, the effect of the filtering will appear as a perturbation to the curvature function whose amplitude is proportional to the amplitude of the transfer function at the sinusoid's frequency.

For curves forming the boundaries of curved objects, zero curvature may, in fact, be an unlikely *a priori* estimate for the curvature since the entire object must have a convex contour at a sufficiently coarse scale, and hence a mean positive curvature. It can be argued that for an curved (object) with area = a the most neutral assumption is that it is a circular blob. In this case, a reasonable default assumption for the mean curvature of such an object would seem to be $1/\sqrt{\pi/a}$.

Regardless of what *a priori* choice we make for natural curvatures of objects, it will be inappropriate for a wide range of structures that are not consistent with it. It is possible, however, to construct a *family* of smoothing functionals $E_i(u_i, c_i) =$

$$\int_{t_a}^{t_b} \|u_i(t) - d(t)\|^2 + \lambda(\kappa(u_i(t), t) - c_i)^2 dt \quad (4)$$

that allow us to extract *multiple* interpretations of the original data as a function of c_i . The minimization of this functional can be performed for various values of i producing different results $u_i(t)$. The parameter c_i tunes the curvature constraint and the minima of the family of functionals specifies the set of functions $u_i(t)$ that correspond to near uniform curvature approximations of the original data. The application of this set of functionals, collectively, will be referred to as **curvature-tuned smoothing (CTS)** and each functional alone as a **layer** of the process. For any functional E_i , the solution function $u_i(t)$ is the smooth approximation of the original data under a different *a priori* assumption, c_i , of what the curvature should be. As such, the constant c_i will be

¹Since the most neutral assumption for the shape of a closed object is a circular blob, if the area is $a = \pi r^2$, then the radius is roughly $\sqrt{a/\pi}$.

referred to as the **target curvature**. In principle, this tuning parameter is a continuous variable that takes on values from $-\infty$ (infinitely concave) through zero (flat) to $+\infty$ (infinitely convex). In practice, we will use this parameter to discretely sample curvature space at fixed intervals. Each minimization for a different value of the target curvature constitutes a **layer** of the CTS process.

This solution expresses a more general analogy to the energy of a rod under tension than the conventional linearized model, but behaves similarly. The case of $c_i \gg 0$ corresponds to an attempt to fit a *convex* solution to the original data, while the solution for $c_i \ll 0$ corresponds to a *concave* fit. The solutions for these different classes of value for c_i may be quite different.

2.1 Surfaces

On a surface, curvature is a two-dimensional property. The *principal curvatures* $\kappa_1(u(x, y), x, y)$ and $\kappa_2(u(x, y), x, y)$ (denoted for simplicity as κ_1 and κ_2) completely define the local curvature structure. As with one-dimensional curvature, these quantities are invariant to rotations and translations of the surface. Furthermore, for several significant classes of surface the principal curvatures are sufficient to recover the original surface.

This allows us to construct the curvature-tuned smoother for surface depth data $d(x, y)$ in three dimensions $E_i(u, c_{1,i}, c_{2,i}) =$

$$\int_S (u(x, y) - d(x, y))^2 + \lambda [(\kappa_1 - c_{1,i})^2 + (\kappa_2 - c_{2,i})^2] dx dy \quad (5)$$

This leads to a two-dimensional descriptive space for object primitives. Curves were described using roughly circular regions with different curvatures. Surfaces are described with patches having different canonical two-dimensional structure: convex or concave spheroidal, cylindrical, etc. A coarse sampling of this space leads to a description akin to that produced using the combinations of signs of curvature. By sampling the two-dimensional curvature space more finely, a description with greater structural resolution is produced. That is, we can discriminate between surface regions that are highly curved cylinders and those that are gently curved cylinders. Examples of the results of the application of this process for specific tuning values are shown in figures 3 and 4.

3 Discontinuities and Parts

The decomposition of the input data is of fundamental importance in obtaining a more concise and abstract description. We do this through the introduction of discontinuities as part of the smoothing and interpolation process. As the smoothing functional converges to its stationary point, the value of the underlying function, analogous to the potential energy density, can be measured locally. This is given by the functional being minimized; for curves: $f(u(t), t) = \|u(t) - d(t)\|^2 + \lambda(\kappa(u(t), t) - c_i)^2$. Its value indicates the degree to which the competing pulls of the smoothness constraint and the underlying data have been simultaneously satisfied at that point.

A variety of elegant methods have been developed to insert discontinuities in a near-optimal manner [14, 5].

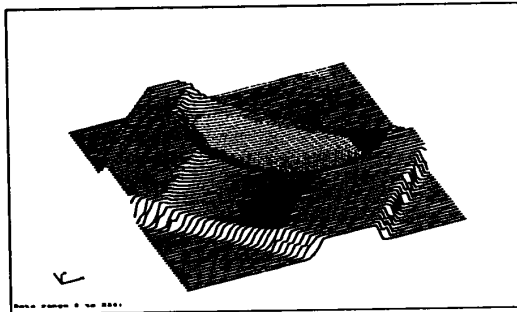


Figure 3: Spherical region extracted from range image. The image depicts a block-shaped part with a concave depression in it. The light lines, show the region that were extracted by a concave spherical tuning.

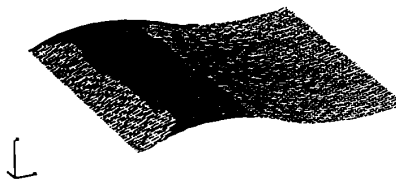


Figure 4: Cylindrical convex region extracted from "wave" image. The image depicts a wavy object from obtain from the University of Pennsylvania GRASP lab range scanner. The dark lines, show the regions that were extracted by a convex tuning.

These sophisticated solutions to the discontinuity localization process could be applied to the curvature tuned smoothing problem. For the application of curvature tuned smoothing described here, we require only the detection of major curve subparts or surface patches and not their precise boundaries. Since the interaction between discontinuities drops off rapidly with distance, we can extract discontinuities in an efficient "greedy" manner if we are willing to accept errors in precise discontinuity placement.

4 Uniqueness and local minima

The non-linearity of the curvature function, and hence the non-quadratic nature of the entire curvature-tuned smoothing functional, renders the minimization *potentially* non-convex even without the discontinuity process. The desirability of a viewpoint invariant functional based on curvature, and its inherent non-linearity, has been observed by other authors [4, 5].

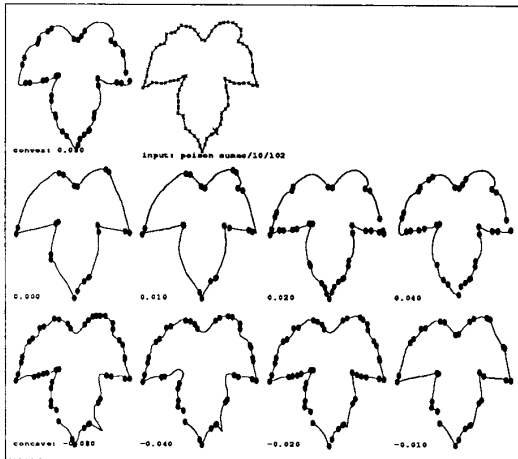


Figure 5: A curve with discontinuities at several tunings. The poison sumac plant leaf, shown in the upper right, is processed using the curvature-tuned smoothing method using several values of the tuning parameter. The tuning value is shown at the lower left of each decomposition. The small circles mark the discontinuities. For most tunings, the curve “shatters” into many small segments, except in regions where the tuning allows the curve to be described in a natural manner.

Although the curvature-tuned smoother, even when tuned to zero curvature, does not satisfy the necessary conditions for a globally convex problem, the existence of local minima is not a difficulty. For small values of λ the stabilizing functional $(\kappa(t) - c_i)^2$ can be approximated by $([x''(t)y'(t) - y''(t)x'(t)]/q(t) - c_i)^2$, where $q(t)$ is a constant function that approximates the first derivative, $q(t) \approx \sqrt{x'(t)^2 + y'(t)^2}$. That is, the curve’s arc length remains roughly constant and the changes in the functional are manifested mainly in the more sensitive second derivative terms. This is particularly relevant for small values of λ since in this case the solution, $u(t)$, cannot drift very far from the input data. In this case, the system’s non-linearity is substantially attenuated. More important, perhaps, is the fact that we are interested in solutions where $u(t)$ and $d(t)$ are fairly close together. It appears, empirically, that the minimum for the curvature-tuned smoother is quite stable due to this condition. The behavior would seem to be characterized by the fact that if other local minima exist, they are far enough away from the original data not to be a problem (for example, they might be cases where the arc-length parameter becomes extremely large, hence pushing the overall curvature value to become small).

In experimental trails using different starting values for the process, the final solution for the difference alternative solutions varied by under one per cent [7]. This suggests that local minima in the solution space to the curvature-tuned smoothing problem are not likely to occur in practice. If they do occur, they must have very

small bowls of convergence or they are located very near to one another such as to make their existence of little practical significance since they would be effectively identical. Local solutions with small radii of convergence might have been missed by the randomized validations and would be potentially problematic. Fortunately, they are unlikely to be present since the curvature function is relatively smooth making such isolated islets unlikely. In addition, other research with both numerical simulations and examination of real thin-plate solids with natural non-zero curvatures supports the supposition of the absence of problematic local minima [13].

5 Curvature scale space

The set of decompositions defined by the *curvature-tuned smoothing* operation can also be described as a scale space. The stabilizer $S(\cdot, c)$ or $S(\cdot, c_1, c_2)$ used for curvature tuned smoothing has the property of selecting not only structures that can be naturally described at different curvatures, but also structures with different spatial extents. Low curvature segments are components of circles of large radius. Conversely, the segments selected when the curvature tuning is large must also have large curvatures. This naturally limits both their length and spatial extent. The structure of a curve in the two-dimensional space defined by arc and curvature-scale is shown below.

6 Free Parameters

The curvature-tuned smoothing process has three free parameters (for each curvature tuning layer): the curvature tuning value at which the process will be performed, the constant, λ , determining the amount of smoothing, and the discontinuity insertion threshold (or, alternatively, the cost of a discontinuity). Although these three parameters could be varied independently, their physical interpretations indicate that they are, in fact, coupled. The curvature tuning for the smoothing is related to scale since it specifies the curvature for the parts to be extracted during a single minimization. The stabilizing constant, λ , is also related to spatial scale, through its effect on the region of support for the local influence function. The discontinuity insertion threshold, finally, is related to the fineness of the sampling of curvature space, and hence to the values of c_i at which the decompositions will be performed. In order for the results of the CTS process to be scale independent, we require the characteristics of the process to scale uniformly with radius of curvature.

It is appropriate for the region of support for a curvature estimate to be commensurate with the expected radius of curvature [7]. λ_i at each layer can be chosen in proportion to the curvature tuning since the region of support of the influence function kernel is directly proportional to the stabilizing constant, thus $\lambda_i = A c_i$. Only the constant of proportionality, A , for the entire set must be determined.

The discontinuity threshold, ξ_i , determines the precision of the segments produced at any layer; that is, how faithful they must be to the target curvature in order to be allowable. The value of the discontinuity threshold is naturally associated with the distance between curvature tuning values. Since the curvature sampling need not be uniform, this threshold may take on different values for curvatures greater than the target curvature (ξ_+) and for

those *less* than the target curvature (ξ_-). We have:

$$\xi_{-,i} = B|c_i - c_{i-1}| \quad (6)$$

$$\xi_{+,i} = B|c_i - c_{i+1}| \quad (7)$$

where B is a constant of proportionality. In practice, B has a value of roughly 2.0.

The choices for c_i crucially establish the class of parts that the process will be permitted to extract. The important characteristic, however, is that the discontinuity locations, and hence the description, varies only gradually with the tuning parameter. This stable variation of the solution with c_i is crucial to the robust performance in the face of scale change and deformation of the data. This stable behavior in response to changes in the tuning parameter is a direct result of the stabilizing effect of the smoothness term.

7 Forming descriptions

Once the series of minimizing functions and their discontinuities have been determined, a description of the input is available in terms of “curvature-scale space”. It is immediately apparent that the discontinuity locations themselves exhibit structure in this space. In particular, regions free from discontinuities for one value of the tuning parameter (or scale) tend to be free from discontinuities for adjacent values of the curvature tuning parameter.

We can determine the value of the smoothness functional over each of the segments individually. Because the positions of segments along the curve varies gradually as a function of the curvature tuning, the tuned smoothness, $\int S(u(t), t)dt$, of a segment can easily be compared to that of its neighbors in curvature-scale space. This allows us to select those segments across the space of the tuning parameter that have the best (lowest) tuned-smoothness values, and hence that serve as the most natural and consistent descriptions of the data. These are the segments from our *a priori* set of models that suit the input stimulus. By using only these segments to form our object representation we obtain a description that is compact in terms of the number of primitives.

The parts themselves, being sections of roughly uniform curvature, are very simple yet they carry substantial expressive power as a group. For a generally smoothly curved object such as a typical silhouette, as shown earlier, it can be well described with only some 15 or fewer of the segments. In fact, their structure is simple enough that, in general, we do not need to retain all the internal point locations for each segment. As a coarse description, we can encode these parts in a much simpler form: by their length (l_j), mean position (t_j), and the curvature tuning used to extract them ($c_{i,j}$). This simple encoding will be referred to as the (type one) *segment descriptor* for a segment j :

$$s_j^1 = (t_j, l_j, c_{i,j}). \quad (8)$$

and the set of these for an object, o , constitutes its (first order) description, $S^1(o)$. Observe that the curvature tuning at which the descriptor was extracted serves as a coarse estimate of the mean curvature of the segment at a particular smoothing scale. This estimate of mean curvature is a stable measurement; the averaged curva-

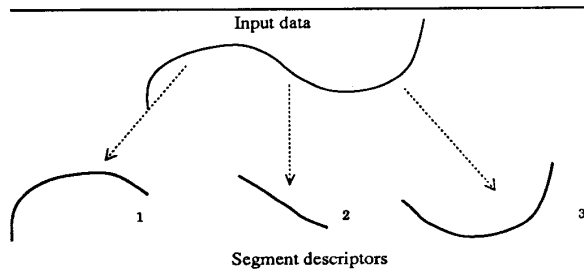


Figure 6: Overlapping segments describing an curve. The segments describing a smooth curve overlap and hence the relative orientations between segments are coded implicitly. In this example, the relative orientation between segments 1 and 3 is constrained by segment 2.

ture over a neighborhood is a far more stable estimator than the local curvature function, even when a second order regularizing smoother is used [9].

For smoothly curved data, this extremely simple description yields substantial disambiguating power. For data that has sharp corners, however, the angle between the corners is not *robustly* described as a smooth curve at any curvature scale. In this case, the description can be augmented by including mean tangent information. For smoothly curved objects this is not necessary since there will be some segment covering every portion of the object and hence the orientation information *between* segments is captured implicitly by the segment that covers the region that joins them. This is illustrated below (figure 6). If there are polygonal objects in the scene, however, it is quite likely that some vertices will have curvature so high (despite the smoothing) that no segment adequately describes them. As a consequence, the representation would provide insufficient information to determine the angle between pairs of straight edges. For example, it would not permit a rectangle and a parallelogram to be distinguished. Although the rich description provided by the encoding above will still be quite good, the use of explicit tangent information may be preferable.

8 Curve Matching

Objects are described by a canonical set of segments (or parts) enumerated as we proceed around the contour. Unlike many other curve matching techniques, these parts express information of varying degrees of locality and embody information at multiple scales at the same location.

To illustrate the sufficiency of the representation we encode the segment using only its ordinal position along the contour, n_j (in other words the relative ordering of the segments), its length, and its curvature tuning:

$$s_j = \langle n_j, l_j, c_{i,j} \rangle. \quad (9)$$

We will perform this matching using structure only in the arc-length dimension, exploiting the structure in the scale dimension only in an implicit manner. That is, a match consists of a sequence of segments along the curve being similar to a sequence from another curve. Since the similarity measure includes the curvature parameters, the scale dimension including effects such as the

containment of a small-scale segment within a coarser one (a small bump within a larger bump) is embedded in the description. The key element in the matching process is the metric for part similarity. To perform the matching, we can readily use a dynamic programming method [12, 8, 1]. As this method implicitly encodes the segment sequence along the contour, we will not directly encode this value in the segment comparison metric.

Segments of low curvature tend to correspond to larger spatial extents of boundary and are computed with a broader influence function and tighter smoothness constraint. For this reason, they are more immune to noise. Furthermore, since we sample curvature space geometrically, low curvature segments are less sensitive to distortions such as those due to perspective or bending; larger absolute changes in radius are required to alter the structure. The segment matching metric has the following form:

$$\langle s_1, s_2 \rangle_s = w_1 |\log c_{i,1} - \log c_{i,2}| + |l_1 - l_2|. \quad (10)$$

As applied to the curve matching problem, the cost function expresses the total cost for the match of one sub-sequence of segments to another. A match is reflected by a path through the cost array. The algorithm used constructs a table of costs such that for two curves composed of type-zero segments $A = s_{a,1}s_{a,2}s_{a,3}\dots s_{a,N}$ and $B = s_{b,1}s_{b,2}s_{b,3}\dots s_{b,N}$. Entry $C(i, j)$ in the cost table has the value of match the first i segments from curve A with the first j segments from curve B . The cost for the diagonal elements in the cost table is thus

$$C(i, i) = \sum_{j=0}^i \langle s_{a,j}, s_{b,j} \rangle_0. \quad (11)$$

The cost array can thus be expressed as

$$C(i, j) = \min \left\{ \begin{array}{l} C(i-1, j) + \langle s_{a,i}, \text{NONE} \rangle_0, \\ C(i, j-1) + \langle \text{NONE}, s_{a,j} \rangle_0, \\ C(i-1, j-1) + \langle s_{a,i}, s_{a,j} \rangle_0 \end{array} \right\} \quad (12)$$

The process of matching one contour with another is then a process of executing the dynamic program for an observed data set against the set of models. The complexity of the dynamic programming process is $O(nm)$. Since the curvature-tuned smoothing description produces a comparatively small set of segments per object this is acceptable (i.e. m is typically around fifteen). A simple method for accelerating the search is to use only the lowest curvature segments (i.e. the coarsest scale) for an initial matching process, followed by a second matching using the full description across curvature space against the subset of models that provided the closest matches.

In experimental trials some fifty object contours, including multiple views of the same object, were processed using the description obtained from curvature smoothing. The distance metric describing similarity of one object to another tended to be broadly distributed among the set of objects: some objects were quite similar to the target, other varied along an axis of increasing dissimilarity. The distance distribution among possible models for one of the objects is shown below (figure 8).

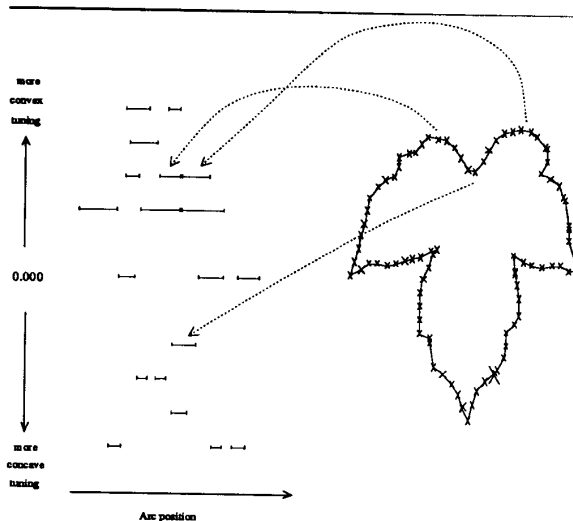


Figure 7: Poison sumac leaf and scale-space. The CTS description of the poison sumac leaf is shown, with the segments corresponding to certain features on the leaf illustrated.

9 Conclusion

The use of a collection of curvature-based minimizing operators, which we term collectively "curvature-tuned smoothing" has been developed to address several difficulties with existing approaches to smoothing, interpolation, segmentation and curve and surface description. The new approach provides a multi-scale description of objects, but one in which the notion of scale is based on curvature-scale space, rather than the conventional definition in terms of Gaussian blurring and spatial frequency. The segmentation and description provided by curvature-tuned smoothing has been shown to provide a natural and powerful vocabulary for the recognition of curved objects. The ease by which this description can be applied has been demonstrated in a dynamic programming context. It shows the ability of the approach to define a broad and continuous range of similarity between objects. This allows objects to be recognized or deemed alike even when they have no identical sub-contours, unlike most existing approaches to curve recognition.

The technique, as presented, has essentially two parameters that must be selected before it is applied. One of these is the sampling grid for curvature space, the other is a scalar value that determines the relevance of the model set. Both of these parameters may be chosen in a manner that allows effective general-purpose operation [7]. On the other hand, these values may also be modified dynamically by higher-level processes allowing interesting possibilities for "attentive" control. For small models sets, the selection of the curvature-space sampling is straightforward. The appropriate selection of a sampling grid in general situations and the refinement of the sampling to obtain greater accuracy dynamically are questions for further investigation.

The method's shortcoming is that it is tied to a model of objects as being composed of circular or quadric regions. While this appears to be appropriate for a wide

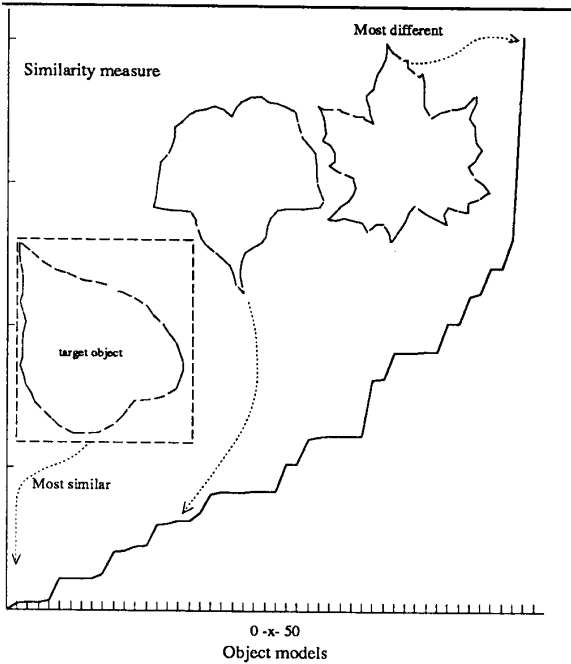


Figure 8: Distribution of model “distances”
The graph illustrates the distribution of similarity measures between different models for the segments extracted from a lilac leaf. This leaf is perceptually very similar to several other leaves in the model set, and hence the distribution is comparatively shallow.

range of natural objects, in particular smoothly curved ones, there are classes of objects for which this descriptive space may be inappropriate (for example, if one wanted to model speech signals or sea urchins).

The advantages of the approach are as follows:

- The extracted descriptors include those that capture the structure of large sections of the data, as well as small sections. By virtue of this multi-scale nature, noise is manifested primarily in the small-scale descriptors and a large measure of robustness is obtained.
- The method can deal with both open and closed curves, and can extract the descriptive primitives from the visible portion of an occluded curve.
- The shape of many natural objects can be described in more than one way. This unavoidable ambiguity is naturally captured by the CTS method. A given region of the input can have more than one “good” descriptor.
- The computations used for computing the description are local in nature. As a result, parallel implementations are realizable.
- In conjunction with the matching process, the experimental results show that the CTS process is effective and realistic.

Several interesting issues remain to be explored further. These include the complete development of CTS-based surface recognition, the inference of volumetric models from the curvature-based segments, and the reconstruction of the original data from the very sparse but rich CTS description.

Acknowledgements

The authors gratefully acknowledge the financial support of Natural Sciences and Engineering Research Council, and Province of Ontario Information Technology Research Center.

References

- [1] Nirwan Ansari and Edward J. Delp. Partial shape recognition: A landmark-based approach. *Pattern Analysis and Machine Intelligence*, 12(5):470–483, May 1990.
- [2] Fred Attneave. Some informational aspects of visual perception. *Psychological Review*, 61:183–193, 1954.
- [3] Paul J. Besl. *Surfaces in Range Image Understanding*. Springer-Verlag, New York, N.Y., 1988.
- [4] Andrew Blake and Andrew Zisserman. Invariant surface reconstruction using weak continuity constraints. *Proceedings of CVPR*, pages 62–66, 1986.
- [5] Andrew Blake and Andrew Zisserman. *Visual Reconstruction*. MIT Press, Cambridge, Mass., 1987.
- [6] Gregory Dudek and John K. Tsotsos. Using curvature information in the decomposition and representation of planar curves. *NATO Advanced Study Institute of Robotics and Active Vision*, July 1989.
- [7] Gregory L. Dudek. *Shape Representation From Curvature*. PhD thesis, Dept. of Computer Science, University of Toronto, Toronto, Canada, December 1990.
- [8] John W. Gorman, O. Robert Mitchell, and Frank P. Kuhl. Partial shape recognition using dynamic programming. *Pattern Analysis and Machine Intelligence*, 10(2):257–266, March 1988.
- [9] Daniel Keren and Michael Werman. Variations on regularization. *Proceedings of the 10th International Conference on Pattern Recognition*, pages 93–97, June 1990.
- [10] J. J. Koenderink and A. J. van Doorn. Photometric invariants related to solid shape. *Optica Acta*, 27(7):981–996, 1980.
- [11] David G. Lowe. Organization of smooth image curves at multiple scales. *Proceedings of the 2nd ICCV*, pages 558–567, Dec. 1988.
- [12] Evangelos Milios. Recovering shape deformation by an extended circular image representation. *Proceedings of the 2nd ICCV*, pages 20–29, Dec. 1988.
- [13] T. R. Tauchert and W. Y. Lu. Large deformation and postbuckling behavior of an initially deformed rod. *Journal of Non-Linear Mechanics*, 22(6):511–520, 1987.
- [14] Demetri Terzopoulos. Regularization of inverse visual problems involving discontinuities. *Pattern Analysis and Machine Intelligence*, 8(4):413–424, 1986.
- [15] Steven W. Zucker, Chantal David, Allan Dobbins, and Lee Iverson. The organization of curve detection: Coarse tangent fields and fine spline coverings. *Proceedings of the 2nd ICCV*, pages 568–577, Dec. 1988.